GENERALIZED WEIERSTRASS-ENNEPER INDUCING, CONFORMAL IMMERSIONS, AND GRAVITY

Robert Carroll* Boris Konopelchenko†

May, 1995

Abstract

Basic quantities related to 2-D gravity, such as Polyakov extrinsic action, Nambu-Goto action, geometrical action, and Euler characteristic are studied using generalized Weierstrass-Enneper (GWE) inducing of surfaces in $\mathbf{R}^3$. Connection of the GWE inducing with conformal immersion is made and various aspects of the theory are shown to be invariant under the modified Veselov-Novikov hierarchy of flows. The geometry of $h\sqrt{g} = 1$ surfaces ($h \sim$ mean curvature) is shown to be connected with the dynamics of infinite and finite dimensional integrable systems. Connections to Liouville-Beltrami gravity are indicated.

1 INTRODUCTION

2-D gravity is one of the most interesting and intriguing toy models of the last decade. It has been studied very intensively starting with the original papers of Polyakov [38]. The variety of different approaches used is rather impressive (see e.g. [15, 22]). One of the approaches consists in the study of Polyakov's surface analogue of the path integral in terms of original continuous surfaces without discretization, triangularization, matrix models, etc. Interesting results in this direction have been obtained recently in [35, 36, 49, 50] where a theory of conformal immersion connected with W gravity in the conformal gauge, strings, and extrinsic geometry has been developed.

*Mathematics Department, University of Illinois, Urbana, IL 61801

†Physics Department, University of Lecce, 73100 Lecce, Italy and Budker Institute of Nuclear Physics, Novosibirsk 90, Russia

In particular, the importance of 2-D surfaces with $h\sqrt{g} = 1$ was demonstrated where h is the mean curvature. An explicit form of the effective action Γ_{eff} for such surfaces was constructed which is a gauge invariant combination of 2-D intrinsic gravity action in light cone gauge Γ_+, geometric action a la Virasoro Γ_-, and extrinsic Polyakov action $\tilde{S}_P$ as in QCD.

In the present paper we propose a different approach based on generalized Weierstrass-Enneper (GWE) inducing. The method of inducing surfaces was developed in [27, 28]. It allows one to generate surfaces in $\mathbf{R}^3$ via simple formulas and to describe their dynamics via $2+1$ dimensional soliton equations. The GWE inducing is a particular case. In this case the integrable dynamics of surfaces is generated by the modified Veselov-Novikov (mVN) hierarchy of equations. We show that GWE inducing is equivalent to the Kenmotsu representation theorem and establish a correspondence with the conformal immersion theory. We express basic quantities of the theory such as Polyakov extrinsic action, Nambu-Goto action, geometrical action, Euler characteristic, etc. in terms of basic quantities of the GWE inducing (two complex variables ψ_1 and ψ_2). For compact orientable surfaces with $h\sqrt{g} = 1$ it is shown that the Polyakov extrinsic action is invariant under the mVN hierarchy of flows. We demonstrate that the surfaces with $h\sqrt{g} = 1$ are induced via the solution of a $1+1$ dimensional Hamiltonian system. In the one dimensional limit this system is a dynamical system with four degrees of freedom which is completely integrable in the Liouville sense. Connections to Liouville-Beltrami gravity are made in relating $\Gamma_{eff} = 0$ (corresponding to fixed Euler characteristic χ) to $\Gamma_\pm$ and $\tilde{S}_P$; then the invariance of extremal $\Gamma_{eff} = 0$ under mVN flows yields a family of extremal surfaces.

2 BACKGROUND

We will give here some information about the differential geometry of surfaces, the method of inducing surfaces and their integrable evolution, and conformal immersion.

2.1 Surfaces in $\mathbf{R}^3$

We consider a surface in the three dimensional Euclidean space $\mathbf{R}^3$ and will denote the local coordinates of the surface by u^1, u^2. The surface can be defined by the formulas (see e.g. [12, 13, 14, 52])

$$\mathbf{X}^i = x^i(u^1, u^2), \ i = 1, 2, 3 \tag{2.1}$$

where $\mathbf{X}^i$ $(i = 1, 2, 3)$ are the coordinates of the variable point of the surface and $x^i(u^1, u^2)$ $(i = 1, 2)$ are scalar functions. The basic characteristics of the surface are

given by the first (Ω_1) and second (Ω_2) fundamental forms

$$\Omega_1 = ds^2 = g_{\alpha\beta} du^\alpha du^\beta; \ \Omega_2 = d_{\alpha\beta} du^\alpha du^\beta \tag{2.2}$$

where $g_{\alpha\beta}$ and $d_{\alpha\beta}$ are symmetric tensors and $\alpha, \ \beta$ take the values $1, 2$. Here and below it is assumed that summation over repeated indices is performed. The quantities $g_{\alpha\beta}$ and $d_{\alpha\beta}$ are calculated by the formulas

$$g_{\alpha\beta} = \frac{\partial \mathbf{X}^i}{\partial u^\alpha} \cdot \frac{\partial \mathbf{X}^i}{\partial u^\beta}; \ d_{\alpha\beta} = \frac{\partial^2 \mathbf{X}^i}{\partial u^\alpha \partial u^\beta} \cdot \mathbf{N}^i \ (\alpha, \beta = 1, 2) \tag{2.3}$$

where $\mathbf{N}^i$ are the components of the normal vector

$$\mathbf{N}^i = (det\, g)^{-\frac{1}{2}} \epsilon^{ikm} \frac{\partial \mathbf{X}^k}{\partial u^1} \frac{\partial \mathbf{X}^m}{\partial u^2} \ (i = 1, 2, 3) \tag{2.4}$$

and ϵ^{ikm} is a totally antisymmetric tensor with $\epsilon^{123} = 1$. The metric $g_{\alpha\beta}$ completely determines the intrinsic properties of the surface. The Gaussian curvature K of the surface is given by the Gauss formula $K = R_{1212}(det\, g)^{-1}$ where $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor.

Extrinsic properties of surfaces are described by the Gaussian curvature K and the mean curvature $2h = g^{\alpha\beta} d_{\alpha\beta}$. Embedding of the surface into $\mathbf{R}^3$ is described both by $g_{\alpha\beta}$ and $d_{\alpha\beta}$ and it is governed by the Gauss-Codazzi eequations

$$\frac{\partial^2 \mathbf{X}^i}{\partial u^\alpha \partial u^\beta} - \Gamma^\gamma_{\alpha\beta} \frac{\partial \mathbf{X}^i}{\partial u^\gamma} - d_{\alpha\beta} \mathbf{N}^i = 0 \tag{2.5}$$

$$\frac{\partial \mathbf{N}^i}{\partial u^\alpha} + d_{\alpha\gamma} g^{\gamma\beta} \frac{\partial \mathbf{X}^i}{\partial u^\beta} = 0 \ (i = 1, 2, 3; \ \alpha, \beta = 1, 2) \tag{2.6}$$

where $\Gamma^\gamma_{\alpha\beta}$ are the Christofel symbols.

Among the global characteristics of surfaces we mention the integral curvature (see e.g. [12, 13, 14, 52])

$$\chi = \frac{1}{2\pi} \int_S K (det\, g)^{\frac{1}{2}} d^2 u \tag{2.7}$$

where K is the Gaussian curvature and the integration in (2.7) is performed over the surface. For compact oriented surfaces

$$\chi = 2(1 - g)$$

where g is the genus of the surface and we will generally assume that surfaces are compact and oriented unless otherwise specified.

Families of parametric curves on the surface form a system of curvilinear local coordinates on the surface. It is often very convenient to use special types of parametric curves on surfaces as coordinates. We will consider in particular minimal lines (curves of zero length). In this case $g_{11} = g_{22} = 0$, i.e.

$$\Omega_1 = 2g_{12}du^1du^2 \tag{2.8}$$

For real surfaces minimal lines are complex and $\Omega_1 = 2\lambda(z,\bar{z})dzd\bar{z}$ where bar means the complex conjugation and λ is a real function. The Gaussian curvature in this case is reduced to $K = (1/g_{12})(\partial^2 \, log(g_{12})/\partial u^1 \partial u^2)$.

2.2 Surface evolution

First we recall the idea of the method of inducing surfaces following [26, 27]. The main idea is to start with a linear PDE $L(\partial_1, \partial_2)\psi = 0$ in two independent variables u^1, u^2 with matrix valued coefficients (ψ is a square matrix). A formal adjoint operator L^* is obtained via $<\phi, \psi>= \int\int du^1du^2Tr(\phi\psi)$ and one has an adjoint equation $L^*(\partial_1, \partial_2)\psi^* = 0$. It follows that

$$\psi^* L\psi - \psi L^* \psi^* = \partial_1 P_1 - \partial_2 P_2 \tag{2.9}$$

where the P_i are bilinear combinations of ψ and ψ^*. Thus for solutions ψ, ψ^* of $L\psi = 0$ and $L^*\psi^* = 0$ one has $\partial_1 P_1^{ik} = \partial_2 P_2^{ik}$. This implies that there exists w^{ik} such that

$$P_1^{ik} = \partial_2 w^{ik}; \;\; P_2^{ik} = \partial_1 w^{ik} \tag{2.10}$$

and the quantities $X^i = \gamma^{ikj}w^{kj}$ (γ^{ikj} are constant) given by quadratures

$$X^i = \gamma^{ikj} \int_\Gamma (P_2^{kj}du^1 + P_1^{kj}du^2) \tag{2.11}$$

do not depend on the curve Γ. Now consider quantities of the type X^i ($i = 1, 2, 3$) as tentative local coordinates of a surface in $\mathbf{R}^3$ induced by L. For example any three linearly independent solutions ψ_i of $L\psi_i = 0$ would induce a tentative surface (for fixed γ^{ikj}). Assume further that the coefficients of L depend on t and satisfy a t dependent equation $M(\partial_t, \partial_1, \partial_2)\psi = 0$ for some linear operator M. Then compatability of the M equation with $L\psi = 0$ provides a nonlinear PDE for the coefficients of L and we also have an evolving family of surfaces - provided of course that the coordinate functions fit together properly to define a surface.

The method of inducing surfaces described above is not completely new. It is in fact the extension of the ideas of Weierstrass and Enneper for construction of minimal

surfaces (surfaces with $h = 0$). The approach of Weierstrass and Enneper is as follows. Let ϕ and ψ be arbitrary functions and define

$$\partial_z w_1 = i(\psi^2 + \phi^2);\ \partial_z w_2 = \psi^2 - \phi^2;\ \partial_z w_3 = -2p\psi;\ X^1 = \Re w_1 = \tag{2.12}$$

$$\Re \int i(\psi^2 + \phi^2)dz;\ X^2 = \Re w_2 = \Re \int (\psi^2 - \phi^2)dz;\ X^3 = \Re w_3 = -\Re \int 2\psi\phi dz$$

Then the X^i define a minimal surface with $z = c$ and $\bar{z} = \hat{c}$ as minimal lines. Note ϕ and ψ are determined via $\partial_{\bar{z}}\psi = 0$, $\partial_{\bar{z}}\phi = 0$. The straightforward generalization of the Weierstrass-Enneper formulas to the case of nonminimal surfaces was given in [26, 27]. We start with the system

$$L\psi = \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix}\psi + \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}\psi = 0 \tag{2.13}$$

with p real and ψ a 2×2 matrix. For $\psi^T =$ transpose ψ one sees that ψ^* satisfies the same equation as ψ^T so $\psi^* = \psi^T$ can be stipulated. Further for $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ there is a constraint $\sigma_2\psi\sigma_2^{-1} = \bar{\psi}$ so a solution of (2.13) has the form $\tilde{\psi} = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$ (we will use $\tilde{\psi}$ now to avoid confusion with ψ of Section 2.3). For X^i real and $g_{\alpha\beta} = 0$ ($\alpha \neq \beta$) one obtains (cf. [26, 27])

$$\partial_z X^1 = i(\psi_2^2 + \bar{\psi}_1^2);\ \partial_{\bar{z}} X^1 = -i(\psi_1^2 + \bar{\psi}_2^2);\ \partial_z X^2 = \bar{\psi}_1^2 - \psi_2^2; \tag{2.14}$$

$$\partial_{\bar{z}} X^2 = \psi_1^2 - \bar{\psi}_2^2; \partial_z X^3 = -2\psi_2\bar{\psi}_1;\ \partial_{\bar{z}} X^3 = -2\psi_1\bar{\psi}_2;$$

$$g_{12} = \partial_z X^i \partial_{\bar{z}} X^i = 2(\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2)^2 = 2det^2\psi;\ d_{12} = 2pdet\psi$$

Further for real $p(z, \bar{z})$ we can write

$$X^1 + iX^2 = 2i \int_\Gamma (\bar{\psi}_1^2 dz' - \bar{\psi}_2^2 d\bar{z}');\ X^1 - iX^2 = \tag{2.15}$$

$$= 2i \int_\Gamma (\psi_2^2 dz' - \psi_1^2 d\bar{z}');\ X^3 = -2 \int_\Gamma (\psi_2\bar{\psi}_1 dz' + \psi_1\bar{\psi}_2 d\bar{z}')$$

where Γ is an arbitrary curve in $\mathbf{C}$ ending at z. Then $\Omega_1 = 4det^2\tilde{\psi}dzd\bar{z}$ and the Gaussian and mean curvatures are

$$K = -det^{-2}\tilde{\psi}(log\, det\tilde{\psi})_{z\bar{z}};\ h = pdet^{-1}\tilde{\psi} \tag{2.16}$$

and consequently the total curvature is

$$\chi = \frac{1}{2\pi} \int_S \int K\sqrt{det\, g}d^2u = \tag{2.17}$$

$$= -\frac{2i}{\pi}\int_S\int dz\wedge d\bar{z}(log\, det\tilde{\psi})_{z\bar{z}} = \frac{2i}{\pi}\int_{\partial S} dz(log\, det\tilde{\psi})_z$$

(cf. remarks after (2.26) below). Hence χ is determined by the asymptotics of ψ_1 and ψ_2. To examine this write (2.13) as (*) $\psi_{1z} = p\psi_2$; $\psi_{2\bar{z}} = -p\psi_1$ and let $p \to 0$ as $|z| \to \infty$. Then $\psi_1 \sim a(\bar{z})$ and $\psi_2 \sim b(z)$ as $|z| \to \infty$ where a and b are arbitrary functions. For solutions of (*) defined by $|\psi_1|^2 \to |z|^n$, $\psi_2 \to 0$ as $|z| \to \infty$ one obtains $\chi = -2n$. Minimal surfaces $\sim p = 0$ and $\psi = \frac{1}{\sqrt{2}}\psi_2$, $\phi = \frac{1}{\sqrt{2}}\bar{\psi}_1$ yields the Weierstrass-Enneper situation. As for time evolution with $u^1 \sim z$, $u^2 \sim \bar{z}$ the simplest nontrivial example is

$$M(\partial_t, \partial_z, \partial_{\bar{z}}) = \partial_t + \partial_z^3 + \partial_{\bar{z}}^3 + 3\begin{pmatrix} 0 & p_z \\ 0 & w \end{pmatrix}\partial_z + 3\begin{pmatrix} \bar{w} & 0 \\ p_{\bar{z}} & 0 \end{pmatrix}\partial_{\bar{z}} + \frac{3}{2}\begin{pmatrix} \bar{w}_{\bar{z}} & 2pw \\ -2p\bar{w} & w_z \end{pmatrix} \quad (2.18)$$

which corresponds to a nonlinear integral equation for p

$$p_t + p_{zzz} + p_{\bar{z}\bar{z}\bar{z}} + 3p_z w + 3p_{\bar{z}}\bar{w} + \frac{3}{2}p\bar{w}_{\bar{z}} + \frac{3}{2}pw_z = 0;\ w_{\bar{z}} = (p^2)_z \quad (2.19)$$

This equation is the first higher equation in the Davey-Stewartson (DS) hierarchy for p, q with $q = -p$ and it can be connected via a (degenerate) Miura type transformation with the Veselov-Novikov NVN-II equation, so one refers to (2.19) as the modified VN (mVN) equation (cf. [6, 26, 27, 29]). The equations for ψ_1 and ψ_2 are given in (3.27).

The hierarchy of integrable PDE associated with the linear problem (LP) (2.13) arises as compatibility conditions of (2.13) with LP's of the form $\psi_t + A_n\psi = 0$; $A_n = \sum_0^n (q_j(u,t)\partial_{u^1}^{2j+1} + r_j(u,t)\partial_{u^2}^{2j+1})$. All members of this mVN hierarchy commute with each other and are integrable by the inverse scattering method. Thus the integrable dynamics of surfaces referred to their minimal lines is induced by the mVN hierarchy via (2.15). For such dynamics one is able to solve the initial value problem for the surface, namely $(g_{\alpha\beta}(z,\bar{z},t=0), d_{\alpha\beta}(z,\bar{z},t=0)) \mapsto (g_{\alpha\beta}(z,\bar{z},t), d_{\alpha\beta}(z,\bar{z},t))$, using the corresponding results for the equations from the mVN hierarchy. This integrable dynamics of surfaces inherits all properties of the mVN hierarchy. Note that the minimal surfaces ($p = 0$) are invariant under such dynamics.

For the 1-D limit one can impose on p, $\tilde{\psi}$ the following constraints $(\partial_z - \partial_{\bar{z}})p = 0$; $(\partial_{\bar{z}} - \partial_z)\tilde{\psi} = 2i\lambda\tilde{\psi}$ (λ real). Then $\tilde{\psi}^*$ ($f^* \sim \bar{f}$) satisfies the same constraints and consequently the X^k are constrained via $(\partial_{\bar{z}} - \partial_z)X^k = 4i\lambda X^k$ ($k = 1,2,3$). Define now real isometric coordinates σ, s via $z = \frac{1}{2}(s - i\sigma)$ to obtain $p = p(s,t)$, $\tilde{\psi} = exp(\lambda\sigma)\chi(s,t)$ and $X^k = exp(2\lambda\sigma)\tilde{X}^k(s,t)$ ($k = 1,2,3$). It follows that $K = 0$ and $K_m = 2p\, exp(-2\lambda\sigma)$. These equations describe a cone type surface generated by the curve with coordinates $\tilde{X}(s,t)$ - i.e. the surface is effectively reduced to a curve with

curvature $p(s,t)$. The linear problem (2.13) is reduced to a 1-D, AKNS type problem for χ with spectral parameter λ, i.e.

$$\partial_s\chi = \begin{pmatrix} i\lambda & p \\ -p & -i\lambda \end{pmatrix}\chi \tag{2.20}$$

and equation (2.19) is converted into the mKdV equation $p_t + 2p_{sss} + 12p^2p_s = 0$. Similarly the higher mVN equations pass to higher order mKdV equations. In this direction note further $(\partial_{\bar{z}} - \partial_z)X^k \cdot (\partial_{\bar{z}} - \partial_z)X^k = -16\lambda^2 X^k X^k$. Via $\partial_z X^k \partial_z X^k = \partial_{\bar{z}} X^k \partial_{\bar{z}} X^k = 0$ and (2.14) one obtains $(2\lambda^2)X^k X^k = det^2\psi$. But $X^k = exp(2\lambda\sigma)\tilde{X}^k$ and $\psi = exp(\lambda\sigma)\chi$ implies then $(2\lambda^2)\tilde{X}^k\tilde{X}^k = det^2\chi$. But for the 1-D constraint above $det\chi =$ constant (say 1) which entails then $\tilde{X}^k\tilde{X}^k = (2\lambda)^{-2}$. Thus the curve with coordinates $\tilde{X}^k(s,t)$ lies on a sphere of radius $1/2\lambda$ (as in [11]). For $\lambda = 0$ one obtains integrable motions of plane curves as in [16, 17, 32, 34]. Note also that (2.11) implies that tangent vectors to the surface will be expressed in terms of bilinear combinations of ψ and ψ^*.

2.3 Conformal immersions

We go next to [35, 36, 49, 50] involving conformal immersions and will sketch some of the results (cf. also [8, 9, 18, 19, 20, 21, 24, 25, 38, 39, 42, 43, 44, 45, 46, 47, 48, 51, 53, 55, 56]). Consider an oriented 2-D surface immersed in $\mathbf{R}^n$, realized as a conformal immersion of a Riemann surface S, i.e. $X : S \to \mathbf{R}^n$. This means the induced metric on S can be written in the form $g_{11} = g_{22};\ g_{12} = g_{21} = 0$. Pick complex local coordinates $z = \xi^1 + i\xi^2$ and $\bar{z} = \xi^1 - i\xi^2$ so $g_{z\bar{z}} = g_{\bar{z}z} \neq 0$ and $g_{zz} = g_{\bar{z}\bar{z}} = 0$. The Grassmannian $G_{2,n}$ of oriented 2-planes in $\mathbf{R}^n$ can be represented by the complex quadric Q_{n-2} in $\mathbf{C}P^{n-1}$ defined by $\sum_1^n w_k^2 = 0,\ w_k \in \mathbf{C}$ where w_k $(k = 1, ..., n)$ are homogeneous coordinates in $\mathbf{C}P^{n-1}$. Writing $w_k = a_k + ib_k$ with $\vec{A} = \{a_k\},\ \vec{B} = \{b_k\},\ Q_{n-2}$ involves $\|\vec{A}\| = \|\vec{B}\|$ with $\vec{A}\cdot\vec{B} = 0$. Then $\vec{A}$, $\vec{B}$ form the basis for an oriented 2-plane in $\mathbf{R}^n$ and an $SO(2)$ rotation of vectors gives rise to the same point $\{exp(i\theta)w_k\}$ in $\mathbf{C}P^{n-1}$. In the conformal gauge above the tangent plane to S spanned by $(\partial_{\xi^1}X^\mu, \partial_{\xi^2}X^\mu)$ corresponds to the point $(\partial_{\xi^1}X^\mu + i\partial_{\xi^2}X^\mu) \sim \partial_{\bar{z}}X^\mu \in Q_{n-2}$. The (conjugate) Gauss may is defined now by $\bar{G}(z) = [\partial_z X]$. Thus $S \to G_{2,n} \simeq Q_{n-2}$ and one looks for a function $\psi(z,\bar{z})$ (to be determined) such that $\partial_z X^\mu = \psi\phi^\mu$ where $\phi^\mu \in Q_{n-2}$ satisfies $\phi^\mu\phi_\mu = 0$ $(\phi : S \to Q_{n-2})$. Note that a map $S \to Q_{n-2}$ corresponds locally to $\phi(z,\bar{z}) = (\phi_1, ..., \phi_n) \in \mathbf{C}^n/0$ satisfying $\sum_1^n \phi_k^2 = 0$ (here $S \to G_{2,n} \sim (\xi^1, \xi^2) \simeq (z,\bar{z}) \to (\partial_{\xi^1}X^\mu, \partial_{\xi^2}X^\mu)$ while $G_{2,n} \simeq Q_{n-2}$ via $(\partial_{\xi^1}X^\mu, \partial_{\xi^2}X^\mu) \simeq (\partial_z X^\mu)$). The line element in S is $ds^2 = \lambda^2|dz|^2$ where $\lambda^2 = 2\|\partial_z X\|^2 = 2|\psi|^2\|\phi\|^2$ $(\|\phi\|^2 = \phi^\mu\bar{\phi}_\mu)$ and the mean curvature vector field of S is $H^\mu = (2/\lambda^2)X^\mu_{z\bar{z}}$ (note ϕ^μ is tangent to S and H^μ is normal). To see this one uses the

Gauss-Codazzi equations in the form (cf. (2.5))

$$\partial_\alpha\partial_\beta X = \Gamma^\gamma_{\alpha\beta}\partial_\gamma X + H^i_{\alpha\beta}N_i;\ \partial_\alpha N_i = -H^i_{\alpha\beta}g^{\beta\gamma}\partial_\gamma X + (N_j\cdot\partial_\alpha N_i)N_j \tag{2.21}$$

where $\Gamma^\gamma_{\alpha\beta}\sim$ affine connection determined by the induced metric $g_{\alpha\beta}$ and $H^i_{\alpha\beta}$ ($i = 1,...,n-2$) are the components of the second fundamental form along the $n-2$ independent normals N_i. Then one notes that in the conformal gauge only Γ^z_{zz} and $\Gamma^{\bar z}_{\bar z\bar z}$ are nonzero and $H = \frac{1}{2}H^{i\alpha}_\alpha N_i$. Assuming ψ exists it can be determined as follows (cf. also [21]). Express H^μ in terms of ψ and ϕ^μ and write

$$(log\psi)_{\bar z} = -\eta;\ V^\mu \equiv \partial_{\bar z}\phi^\mu - \eta\phi^\mu = \bar\psi\|\phi\|^2H^\mu;\ \eta = (\partial_{\bar z}\phi^\mu)\bar\phi^\mu/\|\phi\|^2 \tag{2.22}$$

Here $\|\phi\|^2 = \phi^\mu\bar\phi_\mu$ and V^μ is the normal component of $\partial_{\bar z}\phi^\mu$. Since H^μ and $\|\phi\|^2$ are real, V^μ can be written $V^\mu = exp(i\alpha)R^\mu$ for R^μ real with α the argument of $\bar\psi$ for $\psi = \rho exp(-i\alpha)$. The first two equations in (2.22) are the integrability conditions on the Gauss map (not every $G_{2,n}$ field ϕ^μ forms a tangent plane to a given surface). Now V^μ is a linear combination of $n-2$ unit normals to S and $V^\mu = exp(i\alpha)R^\mu$ so there are n-3 conditions here plus a remaining condition determined by $\Re\eta_z = -(log\rho)_{z\bar z}$ and $\alpha_{z\bar z} = \Im\eta_z$.

Now we concentrate on $\mathbf{R}^3$ although many results for $\mathbf{R}^n$ appear in [35] for example (cf. also [24]). Thus $G_{2,3}\simeq Q_1\simeq \mathbf{C}P^1\simeq S^2$ and the Gauss map can be expressed by a single complex (Kaehler) function $f(z,\bar z)$ via

$$\phi = (1-f^2, i(1+f^2), 2f);\ \phi^\mu\phi_\mu = 0; \tag{2.23}$$

$$\|\phi\|^2 = 2(1+|f|^2)^2;\ or\ via\ N = \frac{1}{1+|f|^2}(f+\bar f, -i(f-\bar f), |f|^2-1)$$

($N\sim$ normal Gauss map) and the integrability condition $\Im\eta_z = \alpha_{z\bar z}$ is given by

$$\Im[(f_{z\bar z}/f_{\bar z}) - (2\bar f f_z/(1+|f|^2))] = 0 \tag{2.24}$$

One obtains then

$$V^\mu = -2f_{\bar z}N^\mu;\ h = H^\mu N_\mu\ or\ H^\mu = hN^\mu;\ \psi = -\bar f_z/h(1+|f|^2)^2 \tag{2.25}$$

(h is the mean curvature scalar). It follows then that $(log\psi)_{\bar z} = -2\bar f f_{\bar z}/(1+|f|^2)$. From this one computes the Euler characteristic ($\chi(g) = 2(1-g)$)

$$2\pi\chi(g) = \int\sqrt{g}Rd^2\xi = 2\int\frac{|f_{\bar z}|^2-|f_z|^2}{(1+|f|^2)^2}\frac{i}{2}dz\wedge d\bar z \tag{2.26}$$

Note here that (2.26) is expressed via globally defined objects whereas (2.17) requires e.g. $det\tilde\psi \neq 0$ or ∞. We will see that for $h\sqrt{g} = 1$ surfaces $det\tilde\psi = (1/2p)$ so,

assuming $p \neq \infty$ at interior points and that p has bounded derivatives $p_z,\ p_{\bar{z}},\ p_{z\bar{z}}$ in the interior, one can only use (2.17) when $p \neq 0$ at interior points. Since this could preclude some interesting situations we will use (2.26) for calculation and refer to this as χ throughout. The Polyakov action (or action induced by external curvature) is

$$\tilde{S}_P = \frac{2}{g_0^2}\int \|H\|^2\sqrt{g}\,d^2\xi = \frac{2}{g_0^2}\int \frac{\|V\|^2}{\|\phi\|^2}\frac{i}{2}dz\wedge d\bar{z} = \frac{4}{g_0^2}\int \frac{|f_{\bar{z}}|^2}{(1+|f|^2)^2}\frac{i}{2}dz\wedge d\bar{z} \tag{2.27}$$

and the Nambu-Goto action is ($S_{NG} = \sigma\int\sqrt{g}d^2\xi$)

$$S_{NG} = \sigma\int|\psi|^2\|\phi\|^2\frac{i}{2}dz\wedge d\bar{z} = 2\sigma\int\frac{|f_{\bar{z}}|^2}{h^2(z,\bar{z})(1+|f|^2)^2}\frac{i}{2}dz\wedge d\bar{z} \tag{2.28}$$

We will be concerned mainly with $\tilde{S}_P$.

A special role is played by surfaces where $h\sqrt{g} = c$. Thus we will introduce a local orthonormal moving frame $\hat{e}_1,\ \hat{e}_2$, and $\hat{N} \sim \hat{e}_3$ where $\hat{e}_1, \hat{e}_2$ are tangent to S and $\hat{N}$ is normal. One can choose e.g.

$$\hat{e}_1 = \frac{1}{1+|f|^2}(1-\frac{1}{2}(f^2+\bar{f}^2), \frac{i}{2}(f^2-\bar{f}^2), f+\bar{f});\ \hat{e}_2 = \tag{2.29}$$

$$\frac{1}{1+|f|^2}(\frac{i}{2}(f^2-\bar{f}^2), 1+\frac{1}{2}(f^2+\bar{f}^2), -i(f-\bar{f}));\ \hat{N} = \frac{1}{1+|f|^2}(f+\bar{f}, -i(f-\bar{f}), |f|^2-1)$$

The structural equations (2.21) take the form

$$\partial_z\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix} = A_z\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix}; \tag{2.30}$$

where A_z is a matrix involving $f,\ \bar{f},\ f_z,\ \bar{f}_z$. There will also be an analogous identical equation involving $\partial/\partial\bar{z}$. Thus $\partial_z\hat{e}_i = (A_z)_{ij}\hat{e}_j,\ \partial_{\bar{z}}\hat{e}_i = (A_{\bar{z}})_{ij}\hat{e}_j$. Then $A_z, A_{\bar{z}}$ are components of a vector $\vec{A}$ in conformal gauge which transforms as a 2-D $SO(3,\mathbf{C})$ gauge field. Under a local gauge transformation $\hat{e}^T \to g\hat{e}^T,\ S \to S'$, and $A \to A'$ where in an obvious notation (dropping the arrows) $A'_\pm = gA_\pm g^{-1} - (\partial_\pm g)g^{-1}$ ($g \in SO(3,\mathbf{C})$). Using the $SO(2)$ degree of freedom involved in choosing the $\hat{e}_i$ to rotate away a component A_{12} of the tangential connection via $g_0(\psi)$ one arrives at

$$A'_z = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}}(\frac{H_{zz}}{\sqrt{g}}+h\sqrt{g})\\ 0 & 0 & \frac{i}{\sqrt{2}}(\frac{-H_{zz}}{\sqrt{g}}+h\sqrt{g})\\ -\frac{1}{\sqrt{2}}(\frac{H_{zz}}{\sqrt{g}}+h\sqrt{g}) & -\frac{i}{\sqrt{2}}(-\frac{H_{zz}}{\sqrt{g}}+h\sqrt{g}) & 0\end{pmatrix}; \tag{2.31}$$

$$A'_{\bar{z}} = \begin{pmatrix} 0 & -i[(log\bar{\psi})_{\bar{z}} + \frac{2f\bar{f}_{\bar{z}}}{1+|f|^2}] & \frac{1}{\sqrt{2}}(H_{\bar{z}\bar{z}} + h) \\ i[(log\bar{\psi})_{\bar{z}} + \frac{2f\bar{f}_{\bar{z}}}{1+|f|^2}] & 0 & \frac{i}{\sqrt{2}}(H_{\bar{z}\bar{z}} - h) \\ -\frac{1}{\sqrt{2}}(H_{\bar{z}\bar{z}} + h) & -\frac{i}{\sqrt{2}}(H_{\bar{z}\bar{z}} - h) & 0 \end{pmatrix}$$

Here one has used $H_{zz} = H^{\mu}_{zz}N^{\mu} = -2f_z\psi$ where $h = -\bar{f}_z/\psi(1+|f|^2)^2$ along with $\sqrt{g} = |\psi|^2\|\phi\|^2 = 2|\psi|^2(1+|f|^2)^2$. This transformation resembles [46, 47] but works at a deeper level since ψ is involved (cf. [36]). Further argument with currents and a gauge fixing leaves T_{zz} unfixed and the condition $h\sqrt{g} = 1$ (or any constant) then singles out a certain class of surfaces (cf. also [46, 47] where light cone gauge is used). In the conformal gauge $\sqrt{g} = exp(\xi)$ where ξ is the Liouville mode. In particular the Polyakov action $\tilde{S}_P$, or extrinsic geometrical action (2.27), can be considered as a gauge fixed form (in conformal gauge) of the action

$$\hat{S}_P = \frac{i}{g_0^2}\int \sqrt{h}h^{\alpha\beta}\frac{\partial_\alpha f\partial_\beta\bar{f}}{(1+|f|^2)^2}\,dz\wedge d\bar{z} \tag{2.32}$$

(this is the same as (2.27) plus terms modulo Euler characteristic as will be shown later). The EM tensor for (2.32) is $T_{zz} = -\partial_z f\partial_z\bar{f}/(1+|f|^2)^2$ so using $H_{zz} = -2f_z\psi,\ h = -\bar{f}_z/\psi(1+|f|^2)^2,\ \sqrt{g} = 2|\psi|^2(1+|f|^2)^2$, we get $T_{zz} = (H_{zz}/\sqrt{g})(h\sqrt{g}) = H_{zz}/\sqrt{g}$ when $h\sqrt{g} = 1$. For constant $h\sqrt{g}$, $(A'_{\bar{z}})_{12}$ in (2.31) becomes $i\partial_{\bar{z}}log(h) = -i\xi_{\bar{z}}$ ($\sqrt{g} = exp(\xi)$) yielding the transformation for the induced metric in Polyakov's 2-D gravity so $H_{\bar{z}\bar{z}} \sim$ induced metric for surfaces of constant $h\sqrt{g}$ while $T_{zz} = H_{zz}/\sqrt{g} \sim$ EM tensor. Finally one notes that $h\sqrt{g}$ = constant surfaces are characterized by $\psi f_{\bar{z}}$ = constant (cf. (3.9)).

Now [49], which begins with a summary of the $\mathbf{R}^3$ situation just discussed, provides further information. Thus first in summary, if one uses $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$ as independent dynamical degress of freedom (independent of the X^μ variables) then the integrability condition $\partial_{\bar{z}}A'_z - \partial_z A'_{\bar{z}} + [A'_z, A'_{\bar{z}}] = 0$ can be rewritten with ξ or directly as an equation of motion

$$\partial_z^3 H_{\bar{z}\bar{z}} = (\partial_{\bar{z}} - H_{\bar{z}\bar{z}}\partial_z - 2(\partial_z H_{\bar{z}\bar{z}}))(H_{zz}/\sqrt{g}) \tag{2.33}$$

Some useful formulas involving $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$ for $h\sqrt{g} = 1$ can now be obtained as follows. Thus from $\partial_z X^\mu = \psi\phi^\mu,\ \phi^\mu = (1-f^2, i(1+f^2), 2f),\ \psi = -\partial_z\bar{f}/h(1+|f|^2)^2$, and $h\sqrt{g} = 1$ one finds (†) $\psi\partial_{\bar{z}}f = -\frac{1}{2}$ while from the Gauss-Codazzi equations (2.21) $H_{\bar{z}\bar{z}} = -2\bar{\psi}\partial_{\bar{z}}\bar{f}$. This plus (†) yields

$$H_{\bar{z}\bar{z}} = \partial_{\bar{z}}\bar{f}/\partial_z\bar{f} \tag{2.34}$$

Further for $h\sqrt{g} = 1$ the integrability condition (2.24) can be simplified via (†) and $(log\psi)_{\bar{z}} = -\eta$ to

$$f_{\bar{z}\bar{z}}/f_{\bar{z}} - 2\bar{f}f_{\bar{z}}/(1+|f|^2) = 0 \tag{2.35}$$

This implies

$$T_{zz} = H_{zz}/\sqrt{g} = \frac{\partial_z^3 \bar{f}}{\partial_z \bar{f}} - \frac{3}{2}(\frac{\partial_z^2 \bar{f}}{\partial_z \bar{f}})^2 = D_z \bar{f} \tag{2.36}$$

Note here that in (2.34) $H_{\bar{z}\bar{z}}$ has the form of a Beltrami coefficient $\mu = \bar{\partial}\bar{f}/\partial\bar{f}$ and T_{zz} is the corresponding Schwartz derivative. Thus an equation (2.33) becomes $\partial^3\mu = \bar{\partial}T_{zz} - \mu\partial T_{zz} - 2(\partial\mu)T_{zz}$. Now one notes also that the independent dynamical fields $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$ can be parametrized in terms of independent Gaussian maps as $H_{\bar{z}\bar{z}} = \partial_{\bar{z}} f_2/\partial_z f_2$ and $H_{zz}/\sqrt{g} = D_z f_1$ (the f_i determine the image of the X^μ in $G_{2,3}$). Then in in [49] an effective action depending on $f_1,\ f_2$ is determined and the equation of motion (2.33) is used to constrain these fields. First one derives an action invariant under Virasoro symmetries (since $h\sqrt{g} = 1$ surfaces have Virasoro symmetry following earlier remarks - cf. here also [38]). The gauge invariant action Γ_{eff} depends on A_z and $A_{\bar{z}}$ (we omit the $'$ now) via parametrizations $A_z = u^{-1}\partial_z u$ and $A_{\bar{z}} = v^{-1}\partial_{\bar{z}} v$. Here u, v are independent elements of the gauge group and this will correspond to taking $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$ as independent of X^μ. Now write (cf. [38])

$$\Gamma_{eff} = \Gamma_-(A_z) + \Gamma_+(A_{\bar{z}}) - \frac{k}{4\pi} Tr \int A_z A_{\bar{z}} dz \wedge d\bar{z} \tag{2.37}$$

where $k = n_f =$ the number of fermions and

$$\Gamma_-(A_z) = \frac{k}{8\pi} Tr \int [(\partial_z u)u^{-1}(\partial_{\bar{z}} u)u^{-1}] d^2\xi + \tag{2.38}$$

$$+\frac{k}{12\pi} Tr \int \epsilon^{abc}(\partial_a u)u^{-1}(\partial_b u)u^{-1}(\partial_c u)u^{-1} d^2\xi dt$$

Then $\Gamma_+(A_{\bar{z}})$ is given by a similar expression with $u \to v$ and the sign changed in the last integral (cf. also [38]. Now take $A_z^+ = H_{zz}/\sqrt{g},\ A_z^- \equiv h\sqrt{g} = 1$, and $A_z^0 = 0$ to get (cf. (2.31) - factors of $i/2$ are being dropped in integration)

$$\Gamma_-(A_z) = S_-(f_1) = \frac{k}{8\pi} \int \frac{\partial_{\bar{z}} f_1}{\partial_z f_1} [\frac{\partial_z^3 f_1}{\partial_z f_1} - 2(\frac{\partial_z^2 f_1}{\partial_z f_1})^2] dz \wedge d\bar{z} \tag{2.39}$$

This corresponds to geometrical action (cf. [1, 2, 3, 8, 9]). Calculation of $\Gamma_+(A_{\bar{z}})$ from (2.31) is not so easy but in light cone gauge an explicit determination is possible, leading to

$$S_+(f_2) = -\frac{k}{8\pi} \int \frac{\partial_z^2 f_2}{\partial_z f_2} (\frac{\partial_z \partial_{\bar{z}} f_2}{\partial_z f_2} - \frac{\partial_z^2 f_2 \partial_{\bar{z}} f_2}{\partial_z f_2 \partial_z f_2}) dz \wedge d\bar{z} \tag{2.40}$$

This is exactly the form of the light cone action in 2-D intrinsic gravity theory. Finally the total action on $h\sqrt{g} = 1$ surfaces is

$$\Gamma_{eff}(f_1, f_2) = \frac{k}{8\pi} \int \frac{\partial_{\bar{z}} f_1}{\partial_z f_1} [\frac{\partial_z^3 f_1}{\partial_z f_1} - 2(\frac{\partial_z^2 f_1}{\partial_z f_1})^2] dz \wedge d\bar{z}\ - \tag{2.41}$$

$$-\frac{k}{8\pi}\int\frac{\partial_z^2 f_2}{\partial_z f_2}(\frac{\partial_z\partial_{\bar{z}}f_2}{\partial_z f_2}-\frac{\partial_z^2 f_2\partial_{\bar{z}}f_2}{(\partial_z f_2)^2})dz\wedge d\bar{z}-\frac{k}{4\pi}\int\frac{\partial_{\bar{z}}f_2}{\partial_z f_2}D_z f_1 dz\wedge d\bar{z}$$

This is the extrinsic geometric gravitational WZNW action on $h\sqrt{g}=1$ surfaces in light cone gauge. It combines in a gauge invariant way the geometric and light cone action studied in 2-D intrinsic gravity.

The equation of motion for (2.41) is

$$\partial_z^3(\frac{\partial_{\bar{z}}f_2}{\partial_z f_2})-\partial_{\bar{z}}D_z f_1-(\frac{\partial_{\bar{z}}f_2}{\partial_z f_2})\partial_z D_z f_1-2\partial_z(\frac{\partial_{\bar{z}}f_2}{\partial_z f_2})D_z f_1=0 \tag{2.42}$$

obtained by varying f_1 and f_2 independently, and one can see that this is equivalent to (2.33) which can be regarded as relating $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$. It is automatically satisfied when one takes both T_{zz} and $H_{\bar{z}\bar{z}}$ as determined by extrinsic geometry via X^μ. Now one wants an effective action in terms of $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$ through their parametrizations in terms of the f_i such that these fields are independent of X^μ. First one checks that (2.41) is invariant under Virasoro transformations. Next one shows that $\Gamma_{eff}(f_1,f_2)=\Gamma_{eff}(f_1\circ f,f_2\circ f)$ and chooses $f=f_2^{-1}(z,\bar{z})$ where $f_2(f_2^{-1}(z,\bar{z}),\bar{z})=z$ so $\Gamma_{eff}(f_1,f_2)=\Gamma_-(f_1\circ f_2^{-1})=\Gamma_+(f_2\circ f_1^{-1})$ (the last by interchanging f_1,f_2). This leads to

$$\Gamma_+(f_2\circ f_1^{-1})=\Gamma_-(f_1\circ f_2^{-1})=\Gamma_+(f_2)+\Gamma_-(f_1)-\frac{k}{4\pi}\int\frac{\partial_{\bar{z}}f_2}{\partial_z f_2}D_z f_1 dz\wedge d\bar{z} \tag{2.43}$$

Thus in particular the properties of Γ_{eff} can be understood from either $\Gamma_+(f_2\circ f_1^{-1})\sim$ light cone action of intrinsic 2-D gravity (with $f\sim f_2\circ f_1^{-1}$) or from $\Gamma_-(f_1\circ f_2^{-1})\sim$ geometric action arising from quantization of the Virasoro group by coadjoint orbits. The last (coupling) term corresponds exactly to the extrinsic Polyakov action $\tilde{S}_P$ modulo χ (cf. Theorem 4.5). In fact the coupling term in Γ_{eff} is needed in order to make it invariant under Virasoro transformations of F_1, F_2 (recall $H_{\bar{z}\bar{z}}=\mu(F_2)$ and $D_z F_1=T_{zz}=H_{zz}/\sqrt{g}$). Quantization in the $\bar{z}$ sector is developed after modification of the conformal weight of $F_1\circ F_2^{-1}$ (where one is thinking of the geometric action realization). Γ_{eff} is the conformally invariant extension of $\tilde{S}_P$ where T_{zz} and $H_{\bar{z}\bar{z}}$ are the dynamical fields. There is also a hidden Virasoro symmetry on $h\sqrt{g}=1$ surfaces where $H_{\bar{z}\bar{z}}$ and T_{zz} transform as a metric and an energy momentum tensor respectively under Virasoro action. The Gauss map is important in establishing the existence of the Virasoro symmetry in $h\sqrt{g}=1$ surfaces.

2.4 Comments on geometry and gravity

We make here a few further comments about the Liouville equation, Liouville action, etc. The Liouville equation classically involves e.g. $\partial_{z\bar{z}}^2\phi=-\frac{1}{2}Kexp(2\phi)$ or (for

$\rho = exp(2\phi)\ \partial^2_{z\bar{z}} log(\rho) = -K\rho$ where $K \sim$ Gaussian curvature. On the other hand classical conformal unquantized Liouville action involves e.g. $(\gamma \sim \hbar)$

$$S_L = \frac{1}{4\pi\gamma^2}\int\sqrt{\hat{g}}(\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \phi R(\hat{g}) + \frac{\mu}{2}e^{\phi}) \tag{2.44}$$

$$= \frac{1}{4\pi}\int\sqrt{\hat{g}}(\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \frac{1}{\gamma}\phi R(\hat{g}) + \frac{\mu}{2\gamma^2}e^{\gamma\phi})$$

as in [15] (cf. also [9, 22] for other notations). Note that the second formula follows from the first via $\phi \to \gamma\phi$. The equations of motion from (2.44) are evidently

$$\gamma\Delta\phi = R(\hat{g}) + \frac{\mu}{2}e^{\gamma\phi} \tag{2.45}$$

and from [15] $R(exp(2\sigma)\hat{g}) = exp(-2\sigma)[R(\hat{g}) - 2\Delta\sigma)]$. Hence for $\hat{g} \to g = exp(2\sigma)\hat{g}$ and $2\sigma = \gamma\phi$ one has $0 = R(g) + (\mu/2)$ or $R(g) = R(exp(\gamma\phi)\hat{g}) = -(\mu/2)$. Thus the Liouville field ϕ or $\gamma\phi$ is thrown into the metric and one looks for a metric with constant Ricci curvature $-(\mu/2)$. Thus Liouville theory can be thought of as a theory of metrics and and equation such as (2.45) is sometimes called a Liouville equation.

Now we know that the Liouville equation with $g = \hat{g}exp(\gamma\phi)$ provides constant curvature $R_g = -\mu/2$ (given a background metric $\hat{g}$). One has equations of motion of the form $(\gamma = 1)$ $\xi_{z\bar{z}} \sim \Delta\xi = R(\hat{g}) + (\mu/2)exp^{\xi}$ as above. However we must not confuse this with the siutation of [35, 36, 49, 50] where one should emphasize in particular that the Polyakov action of (2.27) or (2.32) is a special action introduced for QCD to cope with quantum fluctuations. It becomes the kinetic energy term of a Grassmannian sigma model (cf. [50, 54] where the Nambu-Goto action or area term also becomes an action with local coupling $1/h^2$). It is not the same as the Polyakov action of Liouville gravity, which is equivalent to the Nambu-Goto action there, but rather a string theoretic term in QCD (as well as a crucial geometric ingredient for W gravity). This is related to the idea that a geometric realization of W gravity as extended 2-D gravity involves, in $\mathbf{R}^3$, surfaces of constant mean curvature density $(h\sqrt{g} = 1)$ in which $T_{zz} \sim (H_{zz}/\sqrt{g})$. The corresponding W algebra in this case is the Virasoro algebra. This is accomplished in a conformal gauge for the induced metric $(\sim H_{\bar{z}\bar{z}})$. The mathematics however, involving the Kaehler function f of (2.23), then leads to formulas similar to those of Liouville-Beltrami intrinsic gravity a la [9, 18, 19, 20, 38, 39, 42, 43, 44, 45, 51] for example (e.g. formulas such as (2.33), (2.39), (2.40), etc.). In particular the Polyakov action $\tilde{S}_P$ or $\hat{S}_P$ leads to the basic EM tensor T_{zz} and metric $H_{\bar{z}\bar{z}}$ above which can be used as basic variables (via Kaehler functions f) in formulating an effective action Γ_{eff} as in (2.41).

Further, following [36], one has to be careful to distinguish conformal gauge and

light cone gauge (cf. here [46, 47] where light cone gauge is used). Also we must recall that in [36], the condition $h\sqrt{g} = 1$ is a gauge fixing, and some formulas hold more generally before such a fixing. For example the Gauss-Codazzi equations imply $H_{\bar{z}\bar{z}} = -2\bar{\psi}\partial_{\bar{z}}\bar{f}$ and in general one has also (cf. equations after (2.32) - this is organized in Section 3).

$$\sqrt{g} = 2|\psi|^2(1+|f|^2)^2;\ H_{zz} = -2f_z\psi;\ h = -\frac{\bar{f}_z}{\psi(1+|f|^2)^2} \tag{2.46}$$

On the other hand after gauge fixing, $h\sqrt{g} = 1$, one has (cf. Section 3)

$$\psi\partial_z f = -\frac{1}{2};\ T_{zz} = H_{zz}/\sqrt{g} \tag{2.47}$$

The formula $T_{zz} = H_{zz}/\sqrt{g}$ arises after gauge fixing but is not itself a fixing (cf. [36, 49]). One notes also that $h\sqrt{g} = 1$ is the conformal analogue of the condition $h = 1$ of [46, 47] where light cone gauge is used with $\sqrt{g} = 1/4$ (in conformal gauge $\sqrt{g} = exp(\xi)$ where ξ is the Liouville mode or field). Similarly in [46, 47] one uses $T_{zz} \sim H_{zz}$. The role of $H_{\bar{z}\bar{z}}$ as induced metric corresponds then (for $h\sqrt{g} = 1$) to $\mu = \bar{\partial}\bar{f}/\partial\bar{f}$ being the induced metric. Equations such as (2.33) take the form then

$$\partial^3\mu = [\bar{\partial} - \mu\partial - 2(\partial\mu)]T_{zz}\ \ (T_{zz} = \frac{H_{zz}}{\sqrt{g}}) \tag{2.48}$$

and as in (2.36) for $h\sqrt{g} = 1$ we have $T_{zz} = D_z\bar{f}$. Such formulas also arise in [18, 19, 20, 42, 43, 44, 45, 51] (cf. [9]) and we will look at this below. We will want to compare now various formulas for various actions involving Beltrami coefficients (divergence terms are frequently added without changing the theory).

3 CONNECTING GWE INDUCING AND CONFORMAL IMMERSIONS

We refer here also to [9] where some preliminary calculations were made.

3.1 Relations between quantities

It is clear that there is a strong interaction between the material just sketched on induced surfaces and conformal immersions; we will establish some precise connections here. This will provide some new relations between integrable systems and gravity theory. First consider (cf. (2.14))

$$\partial_z X^\mu = (i(\psi_2^2 + \bar{\psi}_1^2), \bar{\psi}_1^2 - \psi_2^2, -2\psi_2\bar{\psi}_1) \tag{3.1}$$

Evidently $\partial_z X^\mu \cdot \partial_z X^\mu = 0$ with

$$\|\partial_z X^\mu\|^2 = \partial_z X^\mu \cdot \partial_{\bar z} \bar X^\mu = 2(|\psi_1|^2 + |\psi_2|^2)^2 = 2det^2\tilde\psi \tag{3.2}$$

($\tilde\psi$ will be used for the matrix involving $\psi_1,\ \psi_2$ and ψ will be used in $\partial_z X^\mu = \psi\phi^\mu$). We note that the Weierstrass-Enneper (WE) ideas motivated work of Kenmotsu [25] which underlies some work on the Gauss map (cf. [21]) so there are natural background connections here (some of this is spelled out later). Now let ϕ^μ coordinatize the map $S \to Q_1$ and be represented by (2.23) for some complex function f. We can determine f and ψ in terms of ψ_1, ψ_2 via

$$i(\psi_2^2 + \bar\psi_1^2) = \psi(1-f^2);\ \bar\psi_1^2 - \psi_2^2 = i\psi(1+f^2);\ -2\psi_2\bar\psi_1 = 2\psi f \tag{3.3}$$

This gives

THEOREM 3.1. GWE inducing (2.13), (2.15) and the Kenmotsu representation are equivalent and one has

$$f = i\bar\psi_1/\psi_2;\ \psi = i\psi_2^2 \tag{3.4}$$

Proof. From the formulas

$$(log\psi)_{\bar z} = -\frac{2\bar f f_{\bar z}}{1+|f|^2};\ h = -\frac{\bar f_z}{\psi(1+|f|^2)^2} \tag{3.5}$$

for real h one gets $\bar\psi \bar f_z = \psi f_{\bar z};\ \psi_{\bar z} f = \bar\psi_z \bar f$. The Kenmotsu theorem gives the condition

$$h[f_{z\bar z} - \frac{2\bar f f_z f_{\bar z}}{1+|f|^2}] = h_z f_{\bar z} \tag{3.6}$$

for existence of a surfaces with a given Gauss map and mean curvature h and (3.5) with its conclusion correspond to this (cf. [21], second reference). Writing now

$$\psi_1 = -\bar f\sqrt{-i\bar\psi};\ \psi_2 = \sqrt{-i\psi};\ p = -\frac{\psi f_{\bar z}}{|\psi|(1+|f|^2)} \tag{3.7}$$

one shows that equations (3.5) and its conclusion are equivalent to the system $L\psi = 0$ of (2.13), or $\psi_{1z} = p\psi_2;\ \psi_{2\bar z} = -p\psi_1$. Evidently (3.4) holds and the Jacobian of the transformation $(f,\psi) \to (\bar\psi_1, \psi_2)$ is equal to 2. **QED**

We will now develop some relations between the ψ_i, p, ψ, and f. Situations arising from the constraint $h\sqrt{g} = 1$ will be distinguished from the general case when possible, but the derivations are often run together for brevity. The situations of most interest here involve $h\sqrt{g} = 1$ and we will therefore concentrate on this. First

in general, from (2.16), one has $h = pdet^{-1}\tilde{\psi}$ and $ds^2 = \lambda^2 dzd\bar{z}$ where $\lambda^2 = 2\|\partial_z X^\mu\|^2$ (we choose this definition for λ and will change symbols for other λ). Hence from (3.2)

$$\lambda^2 = 4det^2\tilde{\psi},\ \ det\tilde{\psi} = \lambda/2;\ \ h = 2p/\lambda \tag{3.8}$$

Note also the agreement of K in [26, 35]. Now recall (after (2.32)), $h = -\bar{f}_z/\psi(1+|f|^2)^2$ and $\sqrt{g} = 2|\psi|^2(1+|f|^2)^2$ so $h\sqrt{g} = -2\bar{f}_z|\psi|^2/\psi = -2\bar{f}_z\bar{\psi}$ and since $\overline{(f_z)} = \bar{f}_{\bar{z}}$, one has

$$h\sqrt{g} = 1 \equiv \psi f_{\bar{z}} = -1/2 \tag{3.9}$$

Also for $h\sqrt{g} = 1$ from (2.36) $T_{zz} = H_{zz}/\sqrt{g} = D_z\bar{f}$ and the integrability condition (2.24) takes the form (2.35). This leads to ($\psi_{1z} = p\psi_2 \sim \bar{\psi}_{1\bar{z}} = p\bar{\psi}_2,\ \psi_{2\bar{z}} = -p\psi_1 \sim \bar{\psi}_{2z} = -p\bar{\psi}_1$)

$$\psi f_{\bar{z}} = -\frac{1}{2} = i\psi_2^2(i\bar{\psi}_1/\psi_2)_{\bar{z}} = -\psi_2^2(\frac{\bar{\psi}_{1\bar{z}}}{\psi_2} - \frac{\bar{\psi}_1\psi_{2\bar{z}}}{\psi_2^2}) \tag{3.10}$$

$$= -p(|\psi_1|^2 + |\psi_2|^2) \Rightarrow det\tilde{\psi} = |\psi_1|^2 + |\psi_2|^2 = 1/2p\ \ (h\sqrt{g} = 1)$$

Putting this in $det\tilde{\psi} = \lambda/2$ gives $\lambda = 1/p$ and $h = 2p^2 = 2/\lambda^2$ while $K = -det^{-2}\tilde{\psi}(logdet\tilde{\psi})_{z\bar{z}} = -4p^2(logdet\tilde{\psi})_{z\bar{z}}$ (note $h\sqrt{g} = c$ is of interest here - not $h = c$). We also write ($\bar{f} = -i\psi_1/\bar{\psi}_2,\ \partial_{\bar{z}}\bar{f} = -(i/\bar{\psi}_2^2)(\bar{\psi}_2\psi_{1\bar{z}} - \psi_1\bar{\psi}_{2\bar{z}})$)

$$H_{\bar{z}\bar{z}} = \frac{\bar{f}_{\bar{z}}}{\bar{f}_z} = \frac{\bar{\psi}_2\psi_{1\bar{z}} - \psi_1\bar{\psi}_{2\bar{z}}}{p(|\psi_1|^2 + |\psi_2|^2)} = 2(\bar{\psi}_2\psi_{1\bar{z}} - \psi_1\bar{\psi}_{2\bar{z}}) = 2\bar{\psi}_2^2\partial_{\bar{z}}(\frac{\psi_1}{\bar{\psi}_2})\ \ (h\sqrt{g} = 1) \tag{3.11}$$

Also from (2.36), noting that $\partial_z\bar{f} = -ip(|\psi_1|^2 + |\psi_2|^2)/\bar{\psi}_2^2 = -i/2\bar{\psi}_2^2$, which implies $\partial_z^2\bar{f} = (i/2)2\bar{\psi}_{2z}/\bar{\psi}_2^3 = -ip\bar{\psi}_1/\bar{\psi}_2^3$ and $\partial_z^3\bar{f} = -ip_z\bar{\psi}_1/\bar{\psi}_2^3 - ip\bar{\psi}_{1z}/\bar{\psi}_2^3 - 3ip^2\bar{\psi}_1^2/\bar{\psi}_2^4$, one obtains

$$T_{zz} = \frac{2}{\bar{\psi}_2^2}(p_z\bar{\psi}_1\bar{\psi}_2 + p\bar{\psi}_2\bar{\psi}_{1z}) = \frac{2}{\bar{\psi}_2}\partial_z(p\bar{\psi}_1)\ \ (h\sqrt{g} = 1) \tag{3.12}$$

PROPOSITION 3.2. For $h\sqrt{g} = 1$ we have (3.10) - (3.12).

REMARK 3.3. We indicate here some calculations designed in particular to confirm various results in [36]. Thus for $h\sqrt{g} = 1$ we have first (recall $\overline{(F_z)} = \bar{F}_{\bar{z}}$)

$$f = \frac{i\bar{\psi}_1}{\psi_2};\ \psi = i\psi_2^2;\ \psi_{1z} = p\psi_2;\ \bar{\psi}_{1\bar{z}} = p\bar{\psi}_2;\ \psi_{2\bar{z}} = -p\psi_1;\ \bar{\psi}_{2z} = -p\bar{\psi}_1; \tag{3.13}$$

$$\sqrt{g} = e^\xi = |\psi|^2\|\phi\|^2 = 2|\psi|^2(1+|f|^2)^2 = 2det^2\tilde{\psi} = \frac{\lambda^2}{2};\ \lambda = \frac{1}{p};\ h = 2p^2$$

Recall next that $h\sqrt{g} = 1 \sim \psi f_{\bar{z}} = -1/2$ and from $\sqrt{g} = 2|\psi|^2(1+|f|^2)^2$ one gets $h = 2/\lambda^2 = 2p^2$ (also $h = -\bar{f}_{\bar{z}}/[\psi(1+|f|^2)^2]$ - cf. (3.5)). From $H_{zz} = -2f_z\psi =$

$-(2f_z/f_{\bar z})f_{\bar z}\psi$ and (3.11), namely $H_{\bar z\bar z} = \bar f_{\bar z}/\bar f_z$, we see that $H_{\bar z\bar z} = \overline{(H_{zz})}$. Note that in general one expects only $\overline{(H_{zz})} = \overline{((H_z)_z)} = \overline{(H_z)}_{\bar z} = \bar H_{\bar z\bar z}$. Further from [26] $K = -4p^2(logdet\tilde\psi)_{z\bar z} = -4p^2(log(1/2p))_{z\bar z}$ $(1/2p = |\psi_1|^2 + |\psi_2|^2)$. Evidently $(logh)_{\bar z} = -\xi_{\bar z}$. Further $\sqrt{g} = exp(\xi) = 1/2p^2$ implies $2p^2 = exp(-\xi) = h$ and $\xi = -log(2p^2)$ with $\xi_{z\bar z} = -2(logp)_{z\bar z}$ while $K = -4p^2[log(1/2p)]_{z\bar z}$ implies $K = 2exp(-\xi)(logp)_{z\bar z}$ so $K = 2exp(-\xi)(-\xi_{z\bar z}/2) = -\xi_{z\bar z}exp(-\xi)$ and hence $\xi_{z\bar z} = -Kexp(\xi)$ or $\xi_{z\bar z} = -K/h = -K\sqrt{g}$. Note that in [36] one writes $\xi_{z\bar z} = Kexp(-\xi)$ which is equivalent to $(-\xi)_{z\bar z} = -Kexp(-\xi)$ or $\xi_{z\bar z} = -Kexp(\xi)$. Also we have for $h\sqrt{g} = 1$ the equations $\psi\bar\partial f = -(1/2)$ and this with $h = -(\bar f_z/\psi(1+|f|^2)^2)$ implies $h = (2\partial\bar f\bar\partial f/(1+|f|^2)^2)$ while in general $H_{\bar z\bar z} = -2\bar\psi\bar f_{\bar z}$ plus $h\sqrt{g} = 1$ implies $H_{\bar z\bar z} = \bar\partial\bar f/\partial\bar f$ (cf. (2.34).

We want to exhibit next the restrictions (if any) on p, ψ_1, ψ_2 which are imposed by the requirements (2.28), $h\sqrt{g} = 1$ and $T_{zz} = H_{zz}/\sqrt{g}$ (note (3.12) is the calculation $T_{zz} = D_z\bar f$ and $H_{zz} = -2\psi f_z = -2i\psi_2^2\partial(i\bar\psi_1/\psi_2) = 2(\bar\psi_{1z}\psi_2 - \bar\psi_1\psi_{2z})$). One obtains first then $T_{zz} = H_{zz}/\sqrt{g} = 4p^2(\bar\psi_{1z}\psi_2 - \bar\psi_1\psi_{2z})$ which must equal $(2/\bar\psi_2)\partial(p\bar\psi_1)$ by (3.12). Hence we have the following conditions on p, ψ_1, ψ_2

$$2p^2(\bar\psi_{1z}\psi_2 - \bar\psi_1\psi_{2z}) = \frac{1}{\bar\psi_2}\partial(p\bar\psi_1); \tag{3.14}$$

$$|\psi_1|^2 + |\psi_2|^2 = \frac{1}{2p};\ \psi_{1z} = p\psi_2;\ \psi_{2\bar z} = -p\psi_1$$

(the latter equations being equivalent to $\bar\psi_{1\bar z} = p\bar\psi_2$ and $\bar\psi_{2z} = -p\bar\psi_1$) plus (2.33) (which will turn out not to be a restriction). Recall also

$$\mu = H_{\bar z\bar z} = \frac{\bar\partial\bar f}{\partial\bar f} = -2\bar\psi\bar f_{\bar z} = 2(\bar\psi_2\psi_{1\bar z} - \psi_1\bar\psi_{2\bar z}) \tag{3.15}$$

which leads to $T_{zz} = 2p^2\bar\mu$ which is quite pleasant and equation (2.33) has the form $\partial^3\mu = [\bar\partial - \mu\partial - (2\partial\mu)]T_{zz}$. One can now show with a little calculation that (3.14) and (3.15) are compatible and we have the result

THEOREM 3.4. Given the basic evolving surface equations $\psi_{1z} = p\psi_2$ and $\psi_{2\bar z} = -p\psi_1$ with p real one achieves a fit with comformal immersions via (3.4). The condition $h\sqrt{g} = 1$ implies then that $det\tilde\psi = |\psi_1|^2 + |\psi_2|^2 = (1/2p)$ (and $h = 2p^2$) and these conditions imply the first equation of (3.14) which says that $H_{zz}/\sqrt{g} = T_{zz} = D_z\bar f$. This all implies $T_{zz} = 2p^2\bar\mu$ $(T_{zz} = H_{zz}/\sqrt{g},\ \mu = \bar f_{\bar z}/\bar f_z)$ and the only additional condition then on p, ψ_1, ψ_2 is that (2.33) hold in the form $\partial^3\mu = [\bar\partial - \mu\partial - (2\partial\mu)](2p^2\bar\mu)$. However this equation is always true when $T_{zz} = D_z\bar f$ with $\mu = \bar f_{\bar z}/\bar f_z$ a Beltrami coefficient (as is the case here). This is stated e.g. in [18, 51] and verified in [9] and below in Section 4 (it is also implicit in [8, 10]). This means that (2.33) is

automatically true and hence there are no a priori restrictions on $\psi_i,\ p$ imposed by the fit above, beyond the condition $det\tilde{\psi} = 1/2p$. The Liouville equation $\xi_{z\bar{z}} = -Kexp(\xi)$ also holds automatically here as do the equations (cf. [9, 36]) $\partial\mu + \bar{\partial}\xi = 0$ and $\bar{\partial}\bar{\mu} + \partial\xi = 0$.

Proof: All that remains are the last two equations which arise in [36] when $\sqrt{g} = exp(\xi)$ and the second fundamental form $(H_{\alpha\beta})$ are used as independent variables. We check these as follows. Since $2p^2 = exp(-\xi)$ one has $-\xi = log(2) + 2log(p)$ so the requirement involves $2log(p)_z = \bar{\partial}\mu$ and $2log(p)_{\bar{z}} = \partial\mu$. Then from $\mu = 2(\bar{\psi}_2\psi_{1\bar{z}} - \psi_1\bar{\psi}_{2\bar{z}})$ we get for example

$$\mu_z = 2(\bar{\psi}_{2z}\psi_{1\bar{z}} + \bar{\psi}_2\psi_{1z\bar{z}} - \psi_{1z}\bar{\psi}_{2\bar{z}} - \psi_1\bar{\psi}_{2z\bar{z}} \tag{3.16}$$

$$= 2[-p\bar{\psi}_1\psi_{1\bar{z}} + \bar{\psi}_2(p_{\bar{z}}\psi_2 + p\psi_{2\bar{z}}) - p\psi_2\bar{\psi}_{2\bar{z}} + \psi_1(p_{\bar{z}}\bar{\psi}_1 + p\bar{\psi}_{1\bar{z}})]$$

$$= 2\{p_{\bar{z}}(|\psi_2|^2 + |\psi_1|^2) - p\bar{\psi}_1\psi_{1\bar{z}} - p\psi_2\bar{\psi}_{2\bar{z}} + p[\bar{\psi}_2(-p\psi_1) + \psi_1(p\bar{\psi}_2)]\}$$

Now the last [ ] is zero and from (3.14) we have $\bar{\psi}_1\psi_{1\bar{z}} + \bar{\psi}_{2\bar{z}}\psi_2 = -(p_{\bar{z}}/2p^2)$ which implies $\mu_z = 2log(p)_{\bar{z}}$. The equation $\bar{\mu}_{\bar{z}} = 2log(p)_z$ is then automatic. **QED**

3.2 Expressions and behavior for the actions

We consider next the various actions in terms of the ψ_i. Thus from (2.27) ($f = i\bar{\psi}_1/\psi_2,\ f_{\bar{z}} = ip(|\psi_1|^2 + |\psi_2|^2)/\psi_2^2,\ h\sqrt{g} = 1$)

$$\tilde{S}_P = \frac{2i}{g_0^2}\int\frac{|f_{\bar{z}}|^2}{(1+|f|^2)^2}dz\wedge d\bar{z} = \frac{2i}{g_0^2}\int p^2 dz\wedge d\bar{z} = \frac{i}{2g_0^2}\int\frac{dz\wedge d\bar{z}}{(|\psi_1|^2 + |\psi_2|^2)^2} \tag{3.17}$$

while from (2.39) the geometrical action with $f_1 = \bar{f}$ becomes (cf. calculations in (3.12))

$$S_- = \frac{k}{8\pi}\int\frac{\partial_{\bar{z}}\bar{f}}{\partial_z\bar{f}}[\frac{\partial_z^3\bar{f}}{\partial_z\bar{f}} - 2(\frac{\partial_z^2\bar{f}}{\partial_z\bar{f}})^2]dz\wedge d\bar{z} = \tag{3.18}$$

$$= \frac{k}{4\pi}\int[(\partial_z(p\bar{\psi}_1)/\bar{\psi}_2) - p^2(\bar{\psi}_1/\bar{\psi}_2)^2]dz\wedge d\bar{z}$$

(one notes that calculation with $\bar{f}$ is appropriate since μ, T are defined via $f_1,\ f_2 \sim \bar{f}$). From (2.39) and (3.18) we can write now

$$\frac{\partial_z(p\bar{\psi}_1)}{\bar{\psi}_2} - p^2(\frac{\bar{\psi}_1}{\bar{\psi}_2})^2 = -\frac{\bar{\psi}_{2zz}}{\bar{\psi}_2} - (\frac{\bar{\psi}_{2z}}{\bar{\psi}_2})^2$$

$$= -\partial^2 log\bar{\psi}_2 - 2(\partial log\bar{\psi}_2)^2 \tag{3.19}$$

This leads to

$$S_- = -\frac{k}{4\pi}\int[\partial^2 log\bar{\psi}_2 + 2(\partial log\bar{\psi}_2)^2]dz\wedge d\bar{z} \tag{3.20}$$

We consider also the Nambu-Goto action of (2.28), which we write as

$$S_{NG} = \sigma\int\sqrt{g}d^2\xi = \frac{i\sigma}{2}\int|\psi|^2\|\phi\|^2 dz\wedge d\bar{z} = i\sigma\int|\psi|^2(1+|f|^2)^2 dz\wedge d\bar{z}$$

$$= i\sigma\int(|\psi_1|^2+|\psi_2|^2)^2 dz\wedge d\bar{z} = \frac{i\sigma}{4}\int\frac{dz\wedge d\bar{z}}{p^2} \tag{3.21}$$

Further in general we look at

$$S_+ = -\frac{k}{8\pi}\int\frac{\bar{f}_{zz}}{\bar{f}_z^2}[\bar{f}_{z\bar{z}} - \frac{\bar{f}_{zz}\bar{f}_{\bar{z}}}{\bar{f}_z}]dz\wedge d\bar{z} \tag{3.22}$$

and recall however that $\mu = (\bar{f}_{\bar{z}}/\bar{f}_z)$ so $\mu_z = (\bar{f}_{z\bar{z}}/\bar{f}_z) - (\bar{f}_{\bar{z}}\bar{f}_{zz}/\bar{f}_z^2)$ while $\mu_z = 2(log(p))_{\bar{z}}$ as well. Also

$$\frac{\bar{f}_{zz}}{\bar{f}_z} = \frac{-ip\bar{\psi}_1/\bar{\psi}_2^3}{-i/2\bar{\psi}_2^2} = -2\frac{\bar{\psi}_{2\bar{z}}}{\bar{\psi}_2} = -2\bar{\partial}log\bar{\psi}_2 \tag{3.23}$$

Consequently one has

$$S_+ = \frac{k}{4\pi}\int(log\bar{\psi}_2)_{\bar{z}}(log(p))_{\bar{z}} \tag{3.24}$$

Finally we compute also $\chi(g)$ via (2.26) to get

$$2\pi\chi(g) = \int R\sqrt{g}d^2\xi = i\int p^2[1-|\mu|^2]dz\wedge d\bar{z} \tag{3.25}$$

Thus we can state

THEOREM 3.5 For $h\sqrt{g}=1$ the quantities $\tilde{S}_P,\ S_-,\ S_{NG},\ S_+$, and χ are given via (3.17), (3.20), (3.21), (3.24), and (3.25).

We remark in passing that the genus of our immersed surface corresponds to the degree of the mapping $S\to\mathbf{C}P^1$ and the total curvature is $\chi = 2-2g$ For immersions into $\mathbf{R}^3$ this is the only topological invariant whereas for $\mathbf{R}^4$ one obtains the Whitney self-intersection number, which has an interpretation in QCD (cf. [50]). See here also the discussion in [29] (second book), pp. 169 and 181, in connection with charge and the Ishimori equation, and Remark 5.4.

Consider now the extrinsic Polyakov and Nambu-Goto actions (cf. (2.27) - (3.17) and (2.28) - (3.21)) which we rewrite here as ($h\sqrt{g}=1$)

$$\tilde{S}_P = \frac{2i}{g_0^2}\int p^2 dz\wedge d\bar{z};\ S_{NG} = \frac{i\sigma}{4}\int\frac{dz\wedge d\bar{z}}{p^2} \tag{3.26}$$

Now go to the modified Veselov-Novikov (mVN) equations based on (2.18) to obtain for $M\psi = 0$

$$\psi_{1t} + \psi_{1zzz} + \psi_{1\bar{z}\bar{z}\bar{z}} + 3\bar{w}\psi_{1\bar{z}} + \frac{3}{2}\bar{w}_{\bar{z}}\psi_1 + 3(\frac{\psi_{1z}}{\psi_2})_z\psi_{2z} + 3w\psi_{1z} = 0 \tag{3.27}$$

$$\psi_{2z} + \psi_{2zzz} + \psi_{2\bar{z}\bar{z}\bar{z}} + 3w\psi_{2z} + 3\bar{w}\psi_{2\bar{z}} + \frac{3}{2}w_z\psi_2 + 3(\frac{\psi_{1z}}{\psi_2})_{\bar{z}}\psi_{1\bar{z}} = 0$$

where $w_{\bar{z}} = -[(\psi_{1z}\psi_{2\bar{z}})/(\psi_1\psi_2)]_z$. From the mVN equation (2.19) one has also

$$(p^2)_t + (2p_{zz} - p_z^2 + 3p^2 w)_z + (2pp_{\bar{z}\bar{z}} - p_{\bar{z}}^2 + 3p^2\bar{w})_{\bar{z}} = 0 \tag{3.28}$$

Consequently we obtain (assume a closed surface or zero boundary terms)

$$\frac{d\tilde{S}_P}{dt} = \frac{2i}{g_0^2}\int \frac{\partial(p^2)}{\partial t} dz \wedge d\bar{z} = 0 \tag{3.29}$$

Thus $\tilde{S}_P$ is invariant under the mVN deformations which means there is an infinite family of suraces with the same $\tilde{S}_P$. In particular this would apply to minimal $\tilde{S}_P$ surfaces which in the corresponding quantum problem would correspond to zero modes. Further one knows that the integrals of motion are common for the whole mVN hierarchy (where the n^{th} time variable would correspond e.g. to $M_n \sim \partial_t + \partial_z^{2n+1} + \partial_{\bar{z}}^{2n+1} + \cdots$). In the one dimensional limit this hierarchy is reduced to the mKdV hierarchy. In any event we can state

THEOREM 3.6. For compact oriented surfaces $\tilde{S}_P$ is invariant under the whole mVN hierarchy of deformations ($h\sqrt{g} = 1$).

We note however that separately (3.28) does not yield zero for $\partial_t S_{NG}$ or $\partial_t S_-$.

From the point of view of inducing surfaces one continues to ask what is the role of the condition $h\sqrt{g} = 1$ and this has the following features. Thus consider $\psi_{1z} = p\psi_2$; $\psi_{2\bar{z}} = -p\psi_1$ under the constraint $|\psi_1|^2 + |\psi_2|^2 = 1/2p$, which leads to

$$\psi_{1z} - \frac{1}{2}(\frac{\psi_2}{|\psi_1|^2 + |\psi_2|^2}) = 0;\ \psi_{2\bar{z}} + \frac{1}{2}(\frac{\psi_1}{|\psi_1|^2 + |\psi_2|^2}) = 0 \tag{3.30}$$

This system has several simple properties. First it is Lagrangian wih Lagrangian

$$\mathcal{L} = \psi_1\bar{\psi}_{2z} + \bar{\psi}_1\psi_{2\bar{z}} - \psi_2\bar{\psi}_{1\bar{z}} - \bar{\psi}_2\psi_{1z} + log(|\psi_1|^2 + |\psi_2|^2) \tag{3.31}$$

(confirmation is immediate). Introducing coordinates $z = (x + iy)/2$ one has the system

$$\psi_{1x} - i\psi_{1y} - \frac{1}{2}(\frac{\psi_2}{|\psi_1|^2 + |\psi_2|^2}) = 0;\ \psi_{2x} + i\psi_{2y} + \frac{1}{2}(\frac{\psi_1}{|\psi_1|^2 + |\psi_2|^2}) = 0 \tag{3.32}$$

where x plays the role of time. This system has 4 real integrals of motion, namely

$$C_+ = \int dy(\psi_1^2 + \psi_2^2 + \bar{\psi}_1^2 + \bar{\psi}_2^2);\ C_- = i\int dy(\psi_1^2 + \psi_2^2 - \bar{\psi}_1^2 - \bar{\psi}_2^2); \tag{3.33}$$

$$P = \int dy(\psi_{1y}\bar{\psi}_2 - \bar{\psi}_1\psi_{2y});\ \mathcal{H} = \int dy\{i(\psi_{1y}\bar{\psi}_2 + \psi_{2y}\bar{\psi}_1) + \frac{1}{2}log(|\psi_1|^2 + |\psi_2|^2)\}$$

Again confirmation is straightforward (note $P_x = -(1/2)\int dy\partial_y log(|\psi_1|^2 + |\psi_2|^2)$ and $\mathcal{H}_x = \int dy \cdot 0$). Next we see that the system can be represented in the Hamiltonian form

$$\psi_{1x} = \{\psi_1, \mathcal{H}\};\ \psi_{2x} = \{\psi_2, \mathcal{H}\} \tag{3.34}$$

where the Poisson brackets are given via

$$\{f, g\} = \int dy[\frac{\delta f}{\delta\psi_1}\frac{\delta g}{\delta\bar{\psi}_2} - \frac{\delta f}{\delta\psi_2}\frac{\delta g}{\delta\bar{\psi}_1} - (f \leftrightarrow g)] \tag{3.35}$$

The corresponding symplectic form is $\Omega = d\psi_1 \wedge d\bar{\psi}_2 + d\bar{\psi}_1 \wedge d\psi_2$. The equations (3.34) are easily checked and we omit calculations. One can also say that the interaction part of the Hamiltonian $\mathcal{H}$ is

$$\mathcal{H}_{int} = \frac{1}{2}log\, det\, \tilde{\psi} \tag{3.36}$$

which has a pleasant appearance. Thus (cf. Theorem 3.4)

THEOREM 3.7. For $h\sqrt{g} = 1$ we have (3.30) - (3.36). Thus (3.30) is a Hamiltonian-Lagrangian system inducing surfaces with the property $h\sqrt{g} = 1$ via (2.15).

Let us next consider particular classes of surfaces with $h\sqrt{g} = 1$ which are generated by the Weierstrass-Enneper formulas in the case $p_z = p_{\bar{z}}$ (one dimensional limit corresponding to curves). In this case ($z = (1/2)(x + iy)$) referring to [27] we can write

$$\psi_1 = r(x)e^{\lambda y};\ \psi_2 = s(x)e^{\lambda y} \tag{3.37}$$

where r, s are complex valued functions and $\lambda = i\mu$ is imaginary. The system (3.32) becomes now

$$r_x + \mu r - \frac{1}{2}(\frac{s}{|r|^2 + |s|^2}) = 0;\ s_x - \mu s + \frac{1}{2}(\frac{r}{|r|^2 + |s|^2}) = 0 \tag{3.38}$$

We write $r = r_1 + ir_2,\ s = s_1 + is_2$ then to obtain ($\Xi = r_1^2 + r_2^2 + s_1^2 + s_2^2$)

$$r_{1x} + \mu r_1 - \frac{s_1}{2\Xi} = 0;\ r_{2x} + \mu r_2 - \frac{s_2}{2\Xi} = 0; \tag{3.39}$$

$$s_{1x} - \mu s_1 + \frac{r_1}{2\Xi} = 0;\ s_{2x} - \mu s_2 + \frac{r_2}{2\Xi} = 0$$

It is easily checked that this system has the following two integrals of motion

$$\mathcal{H} = -\mu(r_1 s_1 + r_2 s_2) + \frac{1}{4}log(\Xi);\ M = r_1 s_2 - r_2 s_1 \tag{3.40}$$

Further the system (3.39) is Hamiltonian with

$$r_{ix} = \{r_i, \mathcal{H}\};\ s_{ix} = \{s_i, \mathcal{H}\} \tag{3.41}$$

where the Poisson brackets arise from (3.35) in the form

$$\{f, g\} = \int dy[\frac{\delta f}{\delta r_1}\frac{\delta g}{\delta s_1} - \frac{\delta f}{\delta s_2}\frac{\delta g}{\delta r_2} - (f \leftrightarrow g)] \tag{3.42}$$

One checks that $\mathcal{H}$ and M are in involution ($\{\mathcal{H}, M\} = 0$) and thus the system (3.39) is integrable in the Liouville sense with two degrees of freedom. The induced Weierstrass-Enneper surfaces (developable surfaces generated by the curves) then have the form

$$X^1 + iX^2 = 2ie^{-2i\mu y}\int[(r_1 - ir_2)^2 - (s_1 - is_2)^2]dx';$$

$$X^3 = -2\int(r_1 s_1 + r_2 s_2)dx' - 2My \tag{3.43}$$

and we refer to [31] for more on this. In particular we have (cf. the end of Section 2.2)

THEOREM 3.8. For $h\sqrt{g} = 1$ and $p_z = p_{\bar{z}}$ with $\psi_{\bar{z}} - \psi_z = 2i\lambda\psi,\ \lambda$ real, we obtain (3.37) - (3.43).

4 LIOUVILLE-BELTRAMI GRAVITY

We recall also some results from [18, 51] (we write here T for T_{zz} at times). The presentation in [18, 51] is somewhat abbreviated and unclear at times and we give here an enhanced treatment of this material with proofs in order to utilize some of the results later and also to make propaganda for these matters. We are led to consider the subject as follows. One always will have (2.33) or (2.47) when μ is a Beltrami coefficient and T is the corresponding Schwartzian. If $\mu = \delta H/\delta T$ for some Hamiltonian H then the equation (2.47) for example becomes a Hamiltonian equation $\bar{\partial}T = \{T, H\}$. Such an H can be constructed in light cone gauge in the form of geometric action (cf. Remark 4.2 below). Now it is asserted in [18, 49] that the

coupling term in (2.41) corresponds exactly to the extrinsic Polyakov action $\tilde{S}_P$ and we note that this term has the form

$$S_{int} = -\frac{k}{4\pi}\int \mu(\bar{f})D_z\bar{f}dz \wedge d\bar{z} \tag{4.1}$$

when $f_1 = f_2 = \bar{f}$ (see below for proof). Further following [18, 51] the Polyakov light cone intrinsic action arises by a simple calculation from such an H via use of $\int \mu T_{zz}$ (see Remark 4.2 below). Thus if $ga \sim$ geometric action density and $ipa \sim$ intrinsic Polyakov action, then the connection is $ipa = \mu T - ga$ (cf. below for details). This says that in (2.43) for $f_1 = f_2 = \bar{f}$ the last two terms represent minus the intrinsic Polykov action and the extrinsic Polyakov action is simply the sum of the geometric action and the intrinsic Polyakov action. Actually the corresondence here is precise modulo χ as indicated below. To spell out the details we extract material from [18, 51] as follows. We say more than is needed to display some of the beautiful features of this subject and various connections to KdV are indicated for possible further application to induced curves etc.

Thus we go to [9, 18, 51] and note that $f \sim \bar{f}$ in transferring results to the present context. Let $(z, \bar{z})$ be coordinates on a Riemann surface (RS). A quasiconformal automorphism $z \to f(z,\bar{z}), \bar{z} \to \bar{f}(z,\bar{z})$ is characterized by Beltrami coefficients $\mu_{\bar{z}}^z, \mu_z^{\bar{z}}$ $(\mu = \mu_{\bar{z}}^z)$ defined by

$$\bar{\partial} f - \mu\partial f = 0 \;(\mu = \bar{\partial} f/\partial f) \tag{4.2}$$

The Schwartz derivative (sometimes Schwartzian) is now defined via

$$S(f,z) = \{f,z\} = \frac{\partial^3 f}{\partial f} - \frac{3}{2}(\frac{\partial^2 f}{\partial f})^2 = (log(\partial f)'' - \frac{1}{2}(log(\partial f)'^2 \tag{4.3}$$

and one knows that

$$[\bar{\partial} - \mu\partial - 2(\partial\mu)]S(f,z) = \partial^3\mu \tag{4.4}$$

(direct calculation - cf. [9] for a full evaluation). This is an important result and comes up also in connection with Ur-KdV following [41] - cf. also [38]). Equation (4.4) is also present, but disguised, in [8, 10] for $C = \mu$ and $u \sim T = -\frac{1}{2}\kappa S_f$; the approach here is much more meaningful and revealing. Note here the minus sign in the definition of T; this applies in this section only, and when comparing with the T of Section 3 we will keep matters clear. Equation (4.4) can be rewritten

$$\bar{\partial} T = [-\frac{1}{2}\kappa\partial^3 + 2T\partial + T']\mu;\; T = -\frac{1}{2}\kappa S(f,z) = \kappa(p^2 + p');\; p = -\frac{1}{2}\partial log(\partial f) \tag{4.5}$$

(the distinction between $\partial g = g'$ or $g' + g\partial$ should be clear from context).

We note a few calculations which are needed below. For $z \to f(z) = z + \epsilon(z)$ one has to first order in $\epsilon, \epsilon', ...,$ $S = [(1+\epsilon')\epsilon''' - \frac{3}{2}\epsilon''^2]/(1+\epsilon')^2 \sim \epsilon'''$. The transformation law for T is $T \to f'^2 T(f) + (c/12)S$ in conformal field theory (CFT) so $\Delta T \sim (1+\epsilon')^2(T+\epsilon T') - T + (c/12)\epsilon''' = 2\epsilon' T + \epsilon T' + (c/12)\epsilon'''$ (adjust c and κ to bring notations into correspondence). In (4.9) below in general one needs infinitesimal maps $z \to z + \epsilon(z,\bar{z}), \bar{z} \to \bar{z} + \bar{\epsilon}(z,\bar{z})$ and an expanded transformation law. For such f, $\Delta f \sim f'\Delta z + \dot{f}\Delta\bar{z} \sim f'\epsilon + \mu f'\bar{\epsilon}$ with $f' = 1$ at z ($\dot{f} \sim \partial f/\partial\bar{z}$). Thus, $\Delta f \sim \eta = \epsilon + \mu\bar{\epsilon}$ so $T \to (1+\eta')^2(T+\eta T') - T + (c/12)\eta'''$ is consistent, which will imply (4.9) for $\delta T \sim \Delta T$. As for generators we recall (cf. [7]) that $T_\epsilon = \oint \epsilon(z)T(z)dz$ is called a generator of conformal transformation where $[T_\epsilon, T] \sim \epsilon T' + 2\epsilon' T + (c/12)\epsilon'''$. In our language one can think of

$$T_\eta = \int T(\zeta,\bar{\zeta})\eta(\zeta,\bar{\zeta})d\zeta d\bar{\zeta};\ \{T(z,\bar{z}), T_\eta\} = \tag{4.6}$$

$$\int \{T(z,\bar{z}), T(\zeta,\bar{\zeta})\}\eta(\zeta,\bar{\zeta})d\zeta d\bar{\zeta} = \int\int \delta(\bar{z}-\bar{\zeta})\hat{\mathcal{E}}_{z\bar{z}}\delta(z-\zeta)\eta d\zeta d\bar{\zeta} = \hat{\mathcal{E}}\eta = \delta T$$

as a way of using a generator concept. We refer here also to [5, 8, 10] where some of this information also appears. One can view $\bar{z} \sim t$ and $z \sim x$ for example so (4.5) represents an evolution equation for T (note $f, \bar{f} \sim$ independent variables, $\hat{\mathcal{E}} \sim \tilde{\mathcal{E}} \sim -\lambda\partial^2 + T$, and u of KdV $\sim$ EM tensor as indicated e.g. in [5, 8, 10]).

Now to display this *à* la [18, 51] we assume $\mu = \delta H/\delta T$ for some Hamiltonian H and define the Poisson brackets via

$$\{P(z,\bar{z}), Q(z',\bar{z}')\} = \int d\zeta d\bar{\zeta}(\delta P(z,\bar{z})/\delta T(\zeta,\bar{\zeta}))\hat{\mathcal{E}}(\zeta,\bar{\zeta})(\delta Q(z',\bar{z}')/\delta T(\zeta,\bar{\zeta})) \tag{4.7}$$

where $\hat{\mathcal{E}} = -\frac{1}{2}\kappa\partial^3 + 2T\partial + T'$ (note that $T = T(z,\bar{z})$ in general now, not just $T = T(z)$). For P = T and Q = H with $\mu = \delta H/\delta T$ one gets

$$\{T, H\} = \{P, Q\} = \hat{\mathcal{E}}\mu\ (= \bar{\partial}T) \tag{4.8}$$

which expresses (4.6) in Hamiltonian form for $T = -\frac{1}{2}\kappa S_f$ (note $\delta T(z,\bar{z})/\delta T(\zeta,\bar{\zeta}) = \delta(z-\zeta)\delta(\bar{z}-\bar{\zeta})$). For P = Q = T one has (note $\hat{\mathcal{E}}$ has no $\bar{\partial}$)

$$\{T(z,\bar{z}), T(z',\bar{z}')\} = \delta(\bar{z}-\bar{z}')\hat{\mathcal{E}}\delta(z-z') = \delta(\bar{z}-\bar{z}')(-\frac{1}{2}\kappa\partial^3 + 2T\partial + T')\delta(z-z') \tag{4.9}$$

To interpert this consider a quasiconformal diffeomorphism on the RS: $z \to g(z,\bar{z}), \bar{z} \to \bar{g}(z,\bar{z})$ (these form a quasiconformal group $\mathcal{G}$). The infinitesimal form is $z \to z + \epsilon(z,\bar{z}), \bar{z} \to \bar{z} + \bar{\epsilon}(z,\bar{z})$ and T changes as ($\eta = \epsilon + \mu\bar{\epsilon}$)

$$\delta T = [-\frac{1}{2}\kappa\partial^3 + 2T\partial + T']\eta(z,\bar{z}) \tag{4.10}$$

Then (4.9) means $T(z,\bar{z})$ is the generator of η. Finally

$$\bar{\partial}f = \{f,H\} = \mu\partial f \tag{4.11}$$

so the Beltrami equation is an evolution equation of f. Here one has

$$\{f,H\} = \int (\delta f/\delta T)\hat{\mathcal{E}}\mu d\zeta d\bar{\zeta} = \int (\delta f/\delta T)\bar{\partial}T d\zeta d\bar{\zeta} \Rightarrow \{f,H\} = \bar{\partial}f \tag{4.12}$$

since, via the generating function ideas of (4.6) ($\delta f = \int\{f,T\}\eta d\zeta d\bar{\zeta}$) and the definition (4.7), we have for $\eta = \mu\bar{\epsilon}$ and $\bar{\epsilon} \sim \Delta\bar{z}$

$$\delta f \sim \int \{f(z,\bar{z}), T(\zeta,\bar{\zeta})\}\eta(\zeta,\bar{\zeta})d\zeta d\bar{\zeta} \tag{4.13}$$

$$\sim \int\int \frac{\delta f(z,\bar{z})}{\delta T(\xi,\bar{\xi})}\delta(\bar{\xi}-\bar{\zeta})\hat{\mathcal{E}}_{\xi}\delta(\xi-\zeta)\eta d\xi d\bar{\xi} d\zeta d\bar{\zeta} \sim$$

$$\int \frac{\delta f(z,\bar{z})}{\delta T(\xi,\bar{\xi})}(\hat{\mathcal{E}}\mu)\Delta\bar{z}d\xi d\bar{\xi} \sim \int \frac{\delta f}{\delta T}\bar{\partial}T\Delta\bar{z}d\xi d\bar{\xi}$$

which leads to $\bar{\partial}f = \{f,H\}$.

Now for physics, in the light cone gauge, where $\bar{f}$ plays no role and the EM tensor $\sim$ Schwartzian as above, we can find H such that $\mu = (\delta H/\delta T)$, so (4.8), i.e. $\bar{\partial}T = \{T,H\} = \hat{\mathcal{E}}\mu$, holds. Under an infinitesimal transformation $z \to z+\epsilon(z,\bar{z}), \bar{z} \to \bar{z}+\bar{\epsilon}(z,\bar{z})$ we get formally (here $d^2x \sim (i/2)d\zeta d\bar{\zeta}$)

$$\delta H = \int d^2x(\delta H/\delta T)\delta T = \int d^2x\mu\hat{\mathcal{E}}\eta = \int d^2x\mu\hat{\mathcal{E}}(\delta f/\partial f) \tag{4.14}$$

($\delta f = \eta\partial f$ here). Now integrate (4.14) to get ($\dot{f} \sim \bar{\partial}f$)

$$H = -\frac{\kappa}{4}\int d^2x\frac{\dot{f}}{f'}(\frac{f'''}{f'} - 2\frac{f''^2}{f'^2}) \tag{4.15}$$

which is a multiple of the geometric action of [1, 2]. This integration is not clear in [18, 51] so we will sketch an heuristic derivation as follows. First from (4.14)

$$\delta H = \int d^2x\mu\hat{\mathcal{E}}\eta = \int d^2x\eta\hat{\mathcal{E}}^*\mu = \int d^2x(\frac{\delta f}{f'})[\frac{1}{2}\kappa\partial^3 - 2\partial(T\ \cdot) + T']\mu \tag{4.16}$$

$$= \int d^2x(\frac{\delta f}{f'})[\frac{1}{2}\kappa\mu''' - 2T'\mu - 2T\mu' + T'\mu] = -\int d^2x(\frac{\delta f}{f'})\bar{\partial}T = -\int d^2x(\frac{\dot{T}}{f'})\delta f$$

Hence $(\delta H/\delta f) = -\frac{\dot{T}}{f'}$ (see (4.21) and cf. [9] for computation of $\dot{T}$), and we want an integral for H as in (4.15) where

$$\delta H/\delta f = \sum(-\partial)^n(\delta H/\delta f^{(n)}) - \bar{\partial}(\delta H/\delta\dot{f}) \tag{4.17}$$

in (4.15), where H is identified with the integrand also, in a common abuse of notation. To achieve this we write H in a slightly different form in noting that

$$\partial log(F')\bar{\partial}log(F') = \frac{F''}{F'}\frac{\dot{F}'}{F'} \to -\dot{F}\partial\frac{F''}{F'^2} = -\frac{\dot{F}}{F'}(\frac{F'''}{F'} - 2\frac{F''^2}{F'^2}) \tag{4.18}$$

Here the arrow $\to$ indicates an integration by parts where one assumes F and its derivatives are periodic or vanish suitably at boundaries, or that the region has no boundary (i.e. the region is a compact surface in which case the integrands must represent globally defined objects). In this situation we can rewrite (4.15) as

$$H = \frac{\kappa}{4}\int d^2x\partial log(f')\bar{\partial}log(f') = \frac{\kappa}{4}\int d^2x(\frac{f''\dot{f}'}{f'^2}) \tag{4.19}$$

Then using (4.17) (modulo $\kappa/4$)

$$\delta H/\delta f \sim -\partial(-\frac{2f''\dot{f}'}{f'^3}) + \partial\bar{\partial}(\frac{f''}{f'^2}) + \partial^2(\frac{\dot{f}'}{f'^2}) = 2\frac{\dot{f}'''}{f'^2} - 6\frac{\dot{f}''f''}{f'^3} - 2\frac{\dot{f}'f'''}{f'^3} + 6\frac{\dot{f}'f''^2}{f'^4} \tag{4.20}$$

This is to compare with $-(\frac{\dot{T}}{f'})$ (modulo $\kappa/4$) where

$$-\frac{\dot{T}}{f'} = -(\frac{\kappa}{4})[(\frac{-2}{f'})(\frac{\dot{f}'''}{f'} - \frac{\dot{f}'f'''}{f'^2} - \frac{3}{2}(\frac{2\dot{f}''f''}{f'^2} - \frac{2\dot{f}'f''^2}{f'^3}))] \tag{4.21}$$

Thus one has agreement without further integration by parts. This procedure also gives a natural origin for geometric action, in addition to the Virasoro algebra background of [1, 3, 2, 8]. Let us summarize all this in (cf. [18, 19, 20, 51])

THEOREM 4.1 First one has the equation $\bar{\partial}T = \hat{\mathcal{E}}\mu$ of (4.4) - (4.5). If there exists H such that $\mu = \delta H/\delta T$, one arrives at equations such as (4.8) $\bar{\partial}T = \{T, H\}$, (4.9) $\{T(z,\bar{z}), T(z',\bar{z}')\} = \delta(\bar{z} - \bar{z}')\hat{\mathcal{E}}\delta(z - z')$, and (4.11) $\bar{\partial}f = \{f, H\} = \mu\partial f$. For transformations independent of $\bar{f}$ one can find $\mu = (\delta H/\delta T)$ as in (4.14), (4.16) - (4.21), leading to geometrical action as in (4.15).

REMARK 4.2 Polyakov light cone action arises by a simple calculation from H as in (4.15) for example via (integration by parts - cf. [51] - we assume all terms are globally defined)

$$L = \int \mu T d\zeta d\bar{\zeta} - H = (\frac{\kappa}{4})\int d\zeta d\bar{\zeta}(\frac{\dot{f}'f''}{f'^2} - \frac{\dot{f}f''^2}{f'^3}) \tag{4.22}$$

REMARK 4.3 One can arrive at a number of connections of Liouville-Beltrami theory to KdV following [9, 18, 19, 20, 42, 43, 44, 45, 51]. For example from (4.4) and (4.8) for $\mu = S(f,z)$ one obtains $\bar{\partial}\mu = \mu''' + 3\mu\mu'$ with a background Hamiltonian structure. One can also consider a more general Liouville-Beltrami action

$$S = \int dz \wedge d\bar{z}(\frac{\dot{f}'f''}{f'^2} - \frac{\dot{f}f''^2}{f'^3}) + \int dz \wedge d\bar{z}[\partial\phi(\bar{\partial} - \mu\partial)\phi + \Lambda(e^{\phi} - 1) + 2\phi\partial^2\mu] \quad (4.23)$$

The first term is Polyakov's light cone action for 2-D gravity and the second term is Liouville action with a perturbation $2\phi\partial^2\mu - \mu(\partial\phi)^2$, which is necessary for covariance of the Beltrami-Liouville system. Then such an action coupled with various matter fields leads to KdV equations and bihamiltonian structure.

We return now to the discussion at the beginning of Section 4 and look at $\Xi = S_- + S_{int}$ in (2.41) written as ($f \to \bar{f},\ \mu = \bar{f}_{\bar{z}}/\bar{f}_z$, etc. - we drop a factor $(i/2)$ arising in the z, $\bar{z}$ integration)

$$\Xi = \frac{k}{4\pi}\{\int \frac{\mu}{2}[\frac{\partial_z^3\bar{f}}{\partial_z\bar{f}} - 2(\frac{\partial_z^2\bar{f}}{\partial_z\bar{f}})^2]dz \wedge d\bar{z} - \int \mu[\frac{\partial_z^3\bar{f}}{\partial_z\bar{f}} - \frac{3}{2}(\frac{\partial_z^2\bar{f}}{\partial_z\bar{f}})^2]dz \wedge d\bar{z}\} \quad (4.24)$$

(recall $T_{zz} \sim +S(\bar{f},z)$ in (2.41)). But we see from (4.22) that for $\kappa = k$

$$\frac{1}{2\pi}L = \frac{1}{2\pi}\int \mu T dz \wedge d\bar{z} - H \sim \frac{k}{4\pi}\int \mu[\frac{1}{2}(\frac{\bar{f}'''}{\bar{f}'} - 2(\frac{\bar{f}''}{\bar{f}'})^2) - \quad (4.25)$$

$$-(\frac{\bar{f}'''}{\bar{f}'} - \frac{3}{2}(\frac{\bar{f}''}{\bar{f}'})^2]dz \wedge d\bar{z} = \Xi$$

(where $T \sim -(1/2)\kappa S(\bar{f},z)$ in (4.22)). Further from (4.22) one has then

$$\frac{1}{2\pi}L = \frac{1}{8\pi}\int \frac{\bar{f}_{zz}}{\bar{f}_z}(\frac{\bar{f}_{z\bar{z}}}{\bar{f}_z} - \frac{\bar{f}_{zz}\bar{f}_{\bar{z}}}{\bar{f}_z^2})dz \wedge d\bar{z} = -S_+ \quad (4.26)$$

which reduces $\Gamma_{eff} = S_- + S_{int} + S_+$ to zero (making a suitable extremum). Also $S_+ \sim$ intrinsic Polyakov action so we have a direct calculation showing

THEOREM 4.4. The equation of motion (2.42) is satisfied for $f_1 = f_2 = \bar{f}$ and for this solution $\Gamma_{eff} = 0$.

Now it has been asserted previously that $S_{int} \sim \tilde{S}_P$ and this means of course that $S_{int} \sim \tilde{S}_P$ modulo χ (since from Theorem 3.4 for $h\sqrt{g} = 1$ one has $T_{zz} = 2p^2\bar{\mu}$ and $\mu T_{zz} = 2p^2|\mu|^2 \neq p^2$ - cf. also (2.32)). To make this precise recall from (3.17) and (3.25) that (dropping $(i/2)$ again)

$$\pi\chi \sim \int p^2(1 - |\mu|^2)dz \wedge d\bar{z};\ \tilde{S}_P \sim \frac{4}{g_0^2}\int p^2 dz \wedge d\bar{z} \quad (4.27)$$

Hence in particular, writing $\Upsilon = \frac{g_0^2}{4}\tilde{S}_P - \pi\chi$ we get

$$\Upsilon = \int p^2|\mu|^2 dz \wedge d\bar{z} = \frac{1}{2}\int \mu T = -\frac{2\pi}{k}S_{int} \tag{4.28}$$

THEOREM 4.5. For $h\sqrt{g} = 1$ the interaction term $\int \mu T_{zz} = -(4\pi/k)S_{int}$ above is in fact $\int \mu T = 2\Upsilon = (g_0^2/2)\tilde{S}_P - 2\pi\chi$ and thus is equivalent to the extrinsic Polyakov action modulo χ as stated previously without proof. Note that $\Gamma_{eff} = 0 \sim \chi \neq 0$ generally and one could regard χ as fixed in determining solutions.

REMARK 4.6. In [39] one wants a dimensionless term in the action for string theory and this comes from the extrinsic curvature which is added to S_{NG}. This term is unique (up to divergence terms) and invariant under scale transformations $x \to \lambda x$. It is essential to include this in the action and one is led to a Grassman sigma model with constraint as in (2.24) or (2.35). Such a string theory apparently corresponds to QCD. In [36] one demonstrates that in $\mathbf{R}^3$ the geometry of $h\sqrt{g} = 1$ surfaces is equivalent to WSO(3) gravity but we have not developed this here.

We make also the following observation. Take an extremal $\Gamma_{eff} = 0$ (cf. Theorem 4.3) corresponding to $0 = S_- + S_+ + S_{int}$ with the same $\bar{f}$ ((2.33) is satisfied). Then $\int \mu T \sim S_- + S_+$ (cf. and $\int \mu T \sim \tilde{S}_P$ up to a factor of χ as in Theorem 4.5. We will argue then that preservation of $\tilde{S}_P$ under mVN flows corresponds to preserving the extremal WZNW action Γ_{eff}. Now let us make this more precise. We have (Γ and S will be used interchangeably) $\Gamma_{eff} = S_+ + S_- + S_{int}$ with $\Gamma_{int} = -(k/4\pi)\int \mu T$ (cf. (4.1)). Then via Theorem 4.5, $\int \mu T = 2\Upsilon = (g_0^2/2)\tilde{S}_P - 2\pi\chi$. On the other hand, going to (4.25) we have $(1/2\pi)L = S_- - (k/4\pi)\int \mu T = -S_+$. Hence for common $\bar{f}$ one obtains $S_+ + S_- - (k/4\pi)\int \mu T = 0 =$ extreme Γ_{eff} (recall $\Gamma_{eff} = 0 \sim \chi$ fixed). Now under mVN flows with $h\sqrt{g} = 1$, $\tilde{S}_P$ is preserved, and since the genus g is integer valued, χ will be invariant under continuous deformation. Consequently $\int \mu T$ and $S_+ + S_-$ will also be preserved via the integration (4.22). Hence

THEOREM 4.7 Since $\tilde{S}_P$ is preserved under mVN deformations with $h\sqrt{g} = 1$ via Theorem 5.1, let χ (preserved) be given; then the extremal $\Gamma_{eff} = 0$ equation (for $h\sqrt{g} = 1$) is also preserved, yielding a family of extremal surfaces.

5 MISCELLANEOUS RESULTS

We gather here various additional facts and observations which will be organized as remarks.

REMARK 5.1. Now we recall also that $H_{zz} = \bar{\mu} = f_z/f_{\bar{z}}$ and $H_{\bar{z}\bar{z}} = \mu = \bar{f}_{\bar{z}}/\bar{f}_z$

with $T_{zz} = 2p^2\bar{\mu}$ and μ is given in (3.15). The formula (3.12) for T_{zz} is more useful however since it implies

$$T_{zz} = \frac{2}{\bar{\psi}_2}\partial_z(p\bar{\psi}_1) = -\frac{2\bar{\psi}_{2zz}}{\bar{\psi}_2} \tag{5.1}$$

This says that

$$\bar{\psi}_{2zz} + \frac{T_{zz}}{2}\bar{\psi}_2 = 0 \tag{5.2}$$

which is a Schroedinger equation associated with KdV, with $T_{zz}/2$ playing the role of potential. It is tempting here to think of $(1/2)T_{zz} \sim 2(log\tau)_{zz}$ by analogy to KdV but T_{zz} here is only defined to be a component of an EM tensor. We note from [30] that for geodesic coordinates with $ds^2 = d\xi_1^2 + \mathcal{H}d\xi_2^2$, $\mathcal{H}^2 = G$ (i.e. $\sqrt{g} \sim \mathcal{H}$) with $S_{NG} = \sigma\int\sqrt{g}d^2\xi$, one has

$$\mathcal{H}_{\xi_1\xi_1} + K(\xi_1,\xi_2)\mathcal{H} = 0 \tag{5.3}$$

where $K \sim$ Gaussian curvature. Then $\mathcal{H}$ can be regarded as $\Re\Psi$ where Ψ is a wave function satisfying $(x \sim \xi_1,\ y \sim \xi_2) :\ -\Psi_{xx} + U(x,y)\Psi = \lambda^2\Psi$ for $K = -U + \lambda^2$. There is a wide class of such Ψ so many surfaces could arise related to KdV in this manner. Note that our $H_{\bar{z}\bar{z}}$ and other terms involve conformal gauge but perhaps there is analogous behavior. Note first that in our notation $\sqrt{g} = 1/2p^2$ with $h = 2p^2$ for $h\sqrt{g} = 1$. Thus $\mathcal{H} \sim 1/2p^2$ in some sense, while $H_{\bar{z}\bar{z}} = \mu$ plays the role of induced metric. Here we recall also from Theorem 3.4 that $\partial\mu = -\bar{\partial}\xi \equiv \mu_z = 2(log(p))_{\bar{z}}$ and $\bar{\partial}\bar{\mu} = -\partial\xi \equiv \bar{\mu}_{\bar{z}} = 2(log(p))_z$ while the Liouville equation states that $\xi_{z\bar{z}} = -Kexp(\xi) \sim (log(p))_{z\bar{z}} = K/4p^2$.

REMARK 5.2. There is also another interesting direction suggested by some formulas in [21]. Thus given a map $g : M \mapsto N;\ w = f(z)$, between two surfaces with conformal metrics $ds^2 = \lambda^2|dz|^2$ and $d\sigma^2 = \nu^2|dw|^2$ respectively, one defines an energy density $e(g) = (1/2)\|dg\|^2 = e'(g) + e''(g)$ where $e' \sim (\nu^2/\lambda^2)|w_z|^2$ and $e'' \sim (\nu^2/\lambda^2)|w_{\bar{z}}|^2$. For $D \subset M$ the total energy is $E(g) = \int_D e(g)d^2\xi$ and the associated Euler-Lagrange operator is called the tension field

$$\tau(g) = \frac{4}{\lambda^2}(w_{z\bar{z}} + 2(log\nu)_w w_z w_{\bar{z}}) \tag{5.4}$$

For $N = S^2(R) =$ sphere of radius R and w a conformal parameter obtained by a similarity transformation $S^2(R) \mapsto S^2(1)$ followed by a stereographic projection one has $\nu = 2R/(1+|w|^2)$. Then if M is a plane domain ($\lambda = 1$) one obtains ($w \sim f(z,\bar{z})$)

$$e'(g) = 4R^2\frac{|f_z|^2}{(1+|f|^2)^2};\ e''(g) = 4R^2\frac{|f_{\bar{z}}|^2}{(1+|f|^2)^2} \tag{5.5}$$

$$\tau(g) = 4(f_{z\bar{z}} - \frac{2\bar{f}f_z f_{\bar{z}}}{1+|f|^2}) = 4\ell(f)$$

Setting $F(f) = f_{\bar{z}}/(1+|f|^2)$ and $\hat{F}(f) = f_z/(1+|f|^2)$ we see that g holomorphic $\sim F \equiv 0$, g antiholomorphic $\sim \hat{F} \equiv 0$, and g harmonic $\sim \ell(f) \equiv 0$. This all leads to some interesting relations among the various actions studied earlier. Thus write (h and ah for holomorphic and antiholomorphic respectively)

$$S_{ah} = \int e''(g) d^2\xi;\ S_h = \int e'(g) d^2\xi \tag{5.6}$$

Now via (3.17) we see that (set $d^2\xi = -(2/i) dz \wedge d\bar{z}$)

$$S_{ah} = -4R^2 \frac{2}{i} \int \frac{|f_{\bar{z}}|^2}{(1+|f|^2)^2} dz \wedge d\bar{z} = 4R^2 g_0^2 \tilde{S}_P \tag{5.7}$$

and from (2.26)

$$2\pi\chi = \frac{i}{4R^2} \int (e'' - e') dz \wedge d\bar{z} = \frac{1}{8R^2}(S_{ah} - S_h) \tag{5.8}$$

We note that the notation in [21] corresponds to an interior normal vector so a minus sign adjustment may be needed at times. Further we observe that $df \wedge d\bar{f} = (|f_z|^2 - |f_{\bar{z}}|^2) dz \wedge d\bar{z}$ which leads to

$$2\pi\chi = i \int \frac{df \wedge d\bar{f}}{(1+|f|^2)^2} \tag{5.9}$$

THEOREM 5.3. With no restriction $h\sqrt{g} = 1$ we get (5.7) - (5.9).

REMARK 5.4 One has a direct connection of the context of Remark 5.2 to the classical 2-D $SO(3)$ sigma model following [37, 40]. Here (with appropriate variables and scaling) $S \sim S_h + S_{ah}$ corresponds to the sigma model action and the charge $Q \sim S_h - S_{ah} \sim -\chi$. The equation of motion corresponds then to (using $f \sim w$ in [37] - $\bar{f} \sim w$ could also be used)

$$f_{z\bar{z}} = \frac{2\bar{f} f_z f_{\bar{z}}}{1+|f|^2} \tag{5.10}$$

(cf. (2.24) and (2.35) and note this corresponds to $h_z f_{\bar{z}}/h = 0$ in (3.6)). One finds that $S \geq 4\pi|Q|$ and it turns out that multiinstanton (or antiinstanton) solutions of the form

$$f = \prod_1^k \frac{z - a_j}{z - b_j};\ f = \prod_1^k \frac{\bar{z} - a_j}{\bar{z} - b_j} \tag{5.11}$$

of charge $\pm k$ are the only solutions of (5.10) with finite action.

REMARK 5.5. The formulas (3.17), (3.18) for actions as well as (3.12) and (3.25) for T_{zz} and χ respectively have an interesting structure and one can in fact develop this further. First let us think of ψ_1 and ψ_2 with their conjugates as determining a map $m : (z,\bar{z}) \mapsto (\psi_1,\psi_2) : \mathbf{C} \to \mathbf{C}^2$ (one could also envision $(z,\bar{z}) \mapsto (\psi_1,\bar{\psi}_1,\psi_2,\bar{\psi}_2)$ but this is somewhat less clear). Via the defining equations (*) $\psi_{1z} = p\psi_2,\ \psi_{2\bar{z}} = -p\psi_1$ (and $\bar{\psi}_{1\bar{z}} = p\bar{\psi}_2,\ \bar{\psi}_{2z} = -p\bar{\psi}_1$) one has constraints

$$(\psi_1^2)_z + (\psi_2^2)_{\bar{z}} = 0;\ \frac{\bar{\psi}_{1\bar{z}}}{\bar{\psi}_2} = \frac{\psi_{1z}}{\psi_2};\ \frac{\bar{\psi}_{2z}}{\bar{\psi}_1} = \frac{\psi_{2\bar{z}}}{\psi_1} \tag{5.12}$$

Observe that $\alpha(\psi_1,\psi_2)$ satisfies the same constraints for $\alpha \in \mathbf{C}$, so we can think of $m : \mathbf{C} \to \mathbf{C}P^1 = \mathbf{P}C^1 \simeq S^2$ (Riemann sphere) directly, without using (2.23), (3.1), etc. Now consider the extrinsic Polyakov action $\tilde{S}_P$ of (3.17) in the form $\tilde{S}_P = (2i/g_0^2)\int p^2 dz \wedge d\bar{z}$ $(h\sqrt{g} = 1)$. One can rewrite this in terms of the ψ_i as follows. Using $p = (\psi_{1z}/\psi_2) = (\bar{\psi}_{1\bar{z}}/\bar{\psi}_2)$ one has

$$\tilde{S}_P = \frac{2i}{g_0^2}\int \frac{\psi_{1z}\bar{\psi}_{1\bar{z}}}{|\psi_2|^2} dz \wedge d\bar{z} \tag{5.13}$$

Now use $d\psi_1 = \psi_{1z}dz + \psi_{1\bar{z}}d\bar{z}$ and $d\bar{\psi}_1 = \bar{\psi}_{1z}dz + \bar{\psi}_{1\bar{z}}d\bar{z}$ to get $d\psi_1 \wedge d\bar{\psi}_1 = (\psi_{1z}\bar{\psi}_{1\bar{z}} - \psi_{1\bar{z}}\bar{\psi}_{1z})dz \wedge d\bar{z}$ and write

$$\tilde{S}_P = \frac{2i}{g_0^2}\int \frac{d\psi_1 \wedge d\bar{\psi}_1}{|\psi_2|^2} + \frac{2i}{g_0^2}\int \frac{\psi_{1\bar{z}}\bar{\psi}_{1z}}{|\psi_2|^2} dz \wedge d\bar{z} \tag{5.14}$$

Our basic equations say nothing about $\psi_{1\bar{z}}$ or $\bar{\psi}_{1z}$ however and a change of variables $z \leftrightarrow \bar{z}$ in the second integral does not give a copy of $\tilde{S}_P$ since e.g. $\psi_1(z,\bar{z})_{\bar{z}} \to \psi_1(\bar{z},z)_z \neq \psi_1(z,\bar{z})$ without further hypotheses. Another approach would be to write $p^2 = -(\psi_{1z}/\psi_2)(\psi_{2\bar{z}}/\psi_1) = -(log\psi_1)_z(log\psi_2)_{\bar{z}}$ and then

$$\tilde{S}_P = -\frac{2i}{g_0^2}\int (log\psi_1)_z(log\psi_2)_{\bar{z}} dz \wedge d\bar{z} \tag{5.15}$$

Setting $d(log\psi_1) = (log\psi_1)_z dz + (log\psi_1)_{\bar{z}} d\bar{z}$ and $d(log\psi_2) = (log\psi_2)_z dz + (log\psi_2)_{\bar{z}} d\bar{z}$ we get $d(log\psi_1) \wedge d(log\psi_2) = [(log\psi_1)_z(log\psi_2)_{\bar{z}} - (log\psi_1)_{\bar{z}}(log\psi_2)_z]dz \wedge d\bar{z}$ leading to

$$\tilde{S}_P = -\frac{2i}{g_0^2}\int d(log\psi_1) \wedge d(log\psi_2) - \frac{2i}{g_0^2}\int (log\psi_1)_{\bar{z}}(log\psi_2)_z dz \wedge d\bar{z} \tag{5.16}$$

and again terms $\psi_{1\bar{z}}/\psi_1$ and ψ_{2z}/ψ_2 are not specified a priori.

Suppose now that we require, in addition to the basic equations (*) $\psi_{1z} = p\psi_2,\ \psi_{2\bar{z}} = -p\psi_1$, that

$$\psi_{1\bar{z}} = q\psi_2;\ \psi_{2z} = -q\psi_1 \tag{5.17}$$

for some function $q(z,\bar{z})$. Then $\bar{\psi}_{1z} = \bar{q}\bar{\psi}_2$ and $\bar{\psi}_{2\bar{z}} = -\bar{q}\bar{\psi}_1$ so that all derivatives of the ψ_i would be specified. In particular consider the last term in (5.14) with integrand $(\psi_{1\bar{z}}/\psi_2)(\bar{\psi}_{1z}/\bar{\psi}_2) = |q|^2(z,\bar{z})$ and suppose $|q|^2(z,\bar{z}) = p^2(\bar{z},z)$. Then for $z \leftrightarrow \bar{z}$, $\int |q|^2(z,\bar{z})dz \wedge d\bar{z} \to \int p^2(z,\bar{z})d\bar{z} \wedge dz = -\int p^2(z,\bar{z})dz \wedge d\bar{z}$ and (5.14) yields

$$\tilde{S}_P = \frac{i}{g_0^2}\int \frac{d\psi_1 \wedge d\bar{\psi}_1}{|\psi_2|^2} \tag{5.18}$$

Similarly in (5.17) $(\psi_{1\bar{z}}/\psi_1)(\psi_{2z}/\psi_2) = -q^2(z,\bar{z})$ so if we take q real with $-q^2(z,\bar{z}) = -p^2(\bar{z},z)$ then the last integral in (5.16) becomes $-\int q^2(z,\bar{z})dz \wedge d\bar{z} = -\int p^2(\bar{z},z)dz \wedge d\bar{z} = -\int p^2(z,\bar{z})d\bar{z} \wedge dz = \int p^2(z,\bar{z})dz \wedge d\bar{z}$ so that (5.16) becomes

$$\tilde{S}_P = -\frac{i}{g_0^2}\int d(log\psi_1) \wedge d(log\psi_2) \tag{5.19}$$

If q is real in (5.17) then both (5.18) and (5.19) hold. We can also write from (*) $(\psi_1^2)_z + (\psi_2^2)_{\bar{z}} = 0$ while from (5.17) one has $(\psi_1^2)_{\bar{z}} + (\psi_2^2)_z = 0$. Adding and subtracting leads to

$$(\partial_z + \partial_{\bar{z}})(\psi_1^2 + \psi_2^2) = 0;\ (\partial_z - \partial_{\bar{z}})(\psi_1^2 - \psi_2^2) = 0 \tag{5.20}$$

Hence for (α, β) arbitrary functions

$$\psi_1^2 = \alpha(z+\bar{z}) + \beta(z-\bar{z});\ \psi_2^2 = \alpha(z+\bar{z}) - \beta(z-\bar{z}) \tag{5.21}$$

We note a few additional formulas which hold when (5.17) applies. Thus from (3.15)

$$\mu = H_{\bar{z}\bar{z}} = 2(\bar{\psi}_2\psi_{1\bar{z}} - \psi_1\bar{\psi}_{2\bar{z}}) = 2q(|\psi_1|^2 + |\psi_2|^2) = \frac{q}{p} \tag{5.22}$$

so μ is real, while from (3.14) and Theorem 3.2

$$T_{zz} = 4p^2(\bar{\psi}_{1z}\psi_2 - \bar{\psi}_1\psi_{2z}) = 4p^2q(|\psi_1|^2 + |\psi_2|^2) = 2pq \tag{5.23}$$

If we write (5.22) and (5.23) in terms of the ψ_i one obtains

$$T_{zz} = 2pq = -2\partial log(\psi_1)\partial log(\psi_2);\ H_{\bar{z}\bar{z}} = \frac{q}{p} = -\frac{\bar{\partial}\psi_1^2}{\bar{\partial}\psi_2^2} \tag{5.24}$$

Thus we have proved

THEOREM 5.6 Given (5.17) (plus (*)) one obtains (5.21) for arbitrary (α, β) which can be written as $\psi_1^2 = \alpha(x) + \beta(y)$ and $\psi_2^2 = \alpha(x) - \beta(y)$. Further (given (*)), if (5.17) holds with $q(z,\bar{z}) = p(\bar{z},z)$ real then (5.18) and (5.19) hold for $\tilde{S}_P$. Thus the

lovely formulas (5.18) - (5.19) occur for the particular class of surfaces for which (cf. (2.14))

$$\partial_z X^1 = i(\alpha+\bar{\alpha}+\bar{\beta}-\beta);\ \partial_z X^2 = \bar{\alpha}+\bar{\beta}-\alpha+\beta; \tag{5.25}$$
$$\partial_z X^3 = -2\sqrt{(\alpha-\beta)(\bar{\alpha}+\bar{\beta})};\ g_{12} = 4[|\alpha|^2+|\beta|^2+|\alpha+\beta||\alpha-\beta|]$$

Further one obtains (5.22) - (5.24).

REMARK 5.7. We note in passing also that when (5.17) holds one gets (cf. (5.18))

$$S_{ah} = 8iR^2\int p^2 dz\wedge d\bar{z};\ S_h = 8iR^2\int q^2 dz\wedge d\bar{z} = -S_{ah} \tag{5.26}$$

so that from (2.26)

$$2\pi\chi = \frac{1}{4R^2}S_{ah} = g_0^2\tilde{S}_P \tag{5.27}$$

Also in connection with (5.17) we note that (5.17) implies μ is real from (5.22) while T_{zz} is then automatically real from (5.23). Conversely if μ is real one must have

$$\bar{\psi}_2\psi_{1\bar{z}} - \psi_1\bar{\psi}_{2\bar{z}} = \psi_2\bar{\psi}_{1z} - \bar{\psi}_1\psi_{2z} \tag{5.28}$$

This can happen in at least two ways, namely

$$\frac{\psi_{1\bar{z}}}{\psi_2} = q;\ \frac{\psi_{2z}}{\psi_1} = r\ \ or\ \frac{\psi_{1\bar{z}}}{\bar{\psi}_1} = -\frac{\psi_{2z}}{\bar{\psi}_2} \tag{5.29}$$

with (q,r) real ($r=-q\sim$ (5.17)). We see from $T_{zz}=2p^2\bar{\mu}$ in Theorem 3.4 that T_{zz} is now automatically real with μ. Further from (3.14) and (5.29) one has

$$\bar{\psi}_1\psi_{1\bar{z}} + \bar{\psi}_{2\bar{z}}\psi_2 = \frac{1}{2}\partial(\frac{1}{p}) \Rightarrow \bar{\psi}_1\psi_2(q+r) = \frac{1}{2}\partial(\frac{1}{p}) \tag{5.30}$$

which implies $\bar{\psi}_1\psi_2$ is real. On the other hand μ real implies $\partial\mu = 2\bar{\partial}log(p)$ and $\bar{\partial}\mu = 2\partial log(p)$ which implies $\partial^2 log(p) = \bar{\partial}^2 log(p)$ and consequently

$$log(p) = F(z+\bar{z}) + G(z-\bar{z}) \sim p = f(z+\bar{z})g(z-\bar{z}) \tag{5.31}$$

Apparently however $r=-q$ in (5.29) is not implied by μ real. Thus in particular the condition (5.17) seems to be rather strong.

REMARK 5.8. The quantities in A'_z and $A'_{\bar{z}}$ of (2.31) can be computed for $h\sqrt{g}=1$. Thus in A'_z one has entries $\pm(1/\sqrt{2})(1+T)$ and $\pm(i/\sqrt{2})(1-T)$ with $T = 2p^2\bar{\mu}$ ($\mu = 2\bar{\psi}_2^2(\psi_1/\bar{\psi}_2)_{\bar{z}}$), while in $A'_{\bar{z}}$ we have $\pm(1/\sqrt{2})(\mu+2p^2),\ \pm(1/\sqrt{2})(\mu-2p^2)$, and $\pm i(log\, p^2)_{\bar{z}}$. The components for the $\hat{e}_i$ in (2.30) are

$$\Lambda = \frac{1}{1+|f|^2} = 2p|\psi_2|^2;\ \Lambda(f^2+\bar{f}^2) = -4p\Re(\frac{\psi_1^2\psi_2}{\bar{\psi}_2}); \tag{5.32}$$

$$\Lambda(f^2-\bar{f}^2)=4ip\Im(\frac{\psi_1^2\psi_2}{\bar{\psi}_2});\ \Lambda(f+\bar{f})=4p\Im(\psi_1\psi_2);\ \Lambda(f-\bar{f})=4ip\Re(\psi_1\psi_2)$$

$$\Lambda[1+\frac{1}{2}(f^2+\bar{f}^2)]=2p[|\psi_2|^2-\Re(\frac{\psi_1\psi_2}{\bar{\psi}_2})]$$

References

[1] A. Alekseev and S. Shatashvili, Nucl. Phys. B, 323 (1989), 719-723

[2] H. Aratyn, E. Nissimov, and S. Pacheva, Phys. Lett. B, 251 (1990), 401-405

[3] H. Aratyn, E. Nissimov, S. Pacheva, and A. Zimerman, Phys. Lett. B, 240 (1990), 127-132; 242 (1990), 377-382

[4] J. Baez and J. Muniain, Gauge fields, knots, and gravity, World Scientific, 1994

[5] I. Bakas, Nucl. Phys. B, 320 (1988), 189-203

[6] L. Bogdanov, Teor. Matem. Fizika, 70 (1987), 309-314

[7] R. Carroll, Topics in soliton theory, North-Holland, 1991

[8] R. Carroll, On the philosophy of the spectral variable, Proc. First World Congress of Nonlinear Analysts, deGruyter, 1995, to appear

[9] R. Carroll, KdV, geometry, and gravity, to appear

[10] A. Das and S. Roy, Inter. Jour. Mod. Phys., A, 6 (1991), 1429-1445

[11] A. Doliwa and P. Santini, Phys. Lett. A, 185 (1994), 373-384

[12] M. doCarmo, Riemannian geometry, Birkhauser, 1993

[13] L. Eisenhart, Differential geometry, Princeton, 1947

[14] A. Forsyth, Lectures on the differential geometry of curves and surfaces, Cambridge Univ. Press, 1920

[15] P. Ginsparg and G. Moore, Lectures on 2-D gravity and 2-D string theory, hep-th 9304011

[16] R. Goldstein and D. Petrich, Phys. Rev. Lett., 67 (1991), 3203-3206; 69 (1992), 555-558

[17] R. Goldstein and D. Petrich, Singularities in fluids, plasmas, and optics, Kluwer, 1993, pp. 93-109

[18] H. Guo, Z. Wang, and K. Wu, Phys. Lett. B, 264 (1991), 277-282; 283-291

[19] H. Guo, Z. Wang, K. Wu, and S. Wang, Comm. Theor. Phys., 14 (1990), 99-106

[20] H. Guo, J. Shen, K. Wang, and K. Xu, Jour. Math. Phys., 31 (1990), 2543-2547; Comm. Theor. Phys., 14 (1990), 123-128; Chinese Phys. Lett., 6 (1989), 53-55

[21] D. Hoffman and R. Osserman, Jour. Diff. Geom., 18 (1983), 733-754; Proc. London Math. Soc., 50 (1985), 27-56

[22] E. d'Hoker, Lecture notes on 2-D quantum gravity and Liouville theory, Sixth Swieca Summer School, Brasil, 1991, pp. 282-367

[23] H. Hopf, Lecture Notes Math., 1000, Springer, 1983

[24] Y. Kazama, Y. Satoh, and A. Tsuchiya, hep-th 9409179

[25] K. Kenmotsu, Math. Annalen, 245 (1979), 89-99

[26] B. Konopelchenko, Multidimensional integrable systems and dynamics of surfaces in space, Preprint 1993 (Acad. Sinica, Taipei)

[27] B. Konopelchenko, Budkerinp 93-114, Novosibirsk; Induced surfaces and their integrable dynamics, Stud. Appl. Math., to appear

[28] B. Konopelchenko, Phys. Lett. A, 183 (1993), 153-159

[29] B. Konopelchenko, Introduction to multidimensional integrable equations, Plenum, 1992; Solitons in multidimensions, World Scientific, 1993

[30] B. Konopelchenko, Soliton curvatures of surfaces and spaces, to appear

[31] B. Konopelchenko and I. Taimanov, Constant mean curvature surfaces via integrable dynamical system, to appear

[32] J. Langer and R. Perline, Jour. Nonlin. Sci., 1 (1991), 71-93

[33] F. Lund, Nonlinear equations in mathematics and physics, Reidel, 1978, pp. 143-175

[34] K. Nakayama, H. Segur, and M. Wadati, Phys. Rev. Lett., 69 (1992), 2603-2606

[35] R. Parthasarathy and K. Viswanathan, Inter. Jour. Mod. Phys. A, 7 (1992), 1819-1832

[36] R. Parthasarathy and K. Viswanathan, Inter. Jour. Mod. Phys. A, 7 (1992), 317-337

[37] A. Perelomov, Physica 4D (1981), 1-25

[38] A. Polyakov, Mod. Phys. Lett. A, 2 (1987), 893-898; Phys. Lett. B, 103 (1981), 207-210; Inter. Jour. Mod. Phys. A, 5 (1990), 833-842

[39] A. Polyakov, Nucl. Phys. B, 268 (1986), 406-412

[40] R. Rajaraman, Solitons and instantons, North-Holland, 1987

[41] J. Schiff, hep-th 9205105

[42] J. Shen, Z. Sheng, and K. Li, Phys. Lett. B, 325 (1994), 377-382; Interface between mathematics and physics, World Scientific, 1994, pp. 358-366

[43] J. Shen, Z. Sheng, and Z. Wang, Phys. Lett. B, 291 (1992), 53-62; Interface between mathematics and physics, World Scientific, 1994, pp. 367-373

[44] J. Shen and Z. Sheng, Phys. Lett. B, 279 (1992), 285-290

[45] J. Shen and K. Xu, Comm. Theor. Phys., 11 (1989), 207-214

[46] G. Sotkov, M. Stanishkov, and C. Zhu, Nucl. Phys. B, 356 (1991), 245-268

[47] G. Sotkov and M. Stanishkov, Nucl. Phys. B, 356 (1991), 439-468

[48] L. Takhtajan, New symmetry principles in quantum field theory, Plenum, 1992, pp. 383-406; Topics in quantum geometry of Riemann surfaces; 2-D quantum gravity, SUNY Stony Brook, 1994; Mod. Phys. Lett. A, 1994), 2293-2299

[49] K. Viswanathan and R. Parthasarathy, A conformal field theory of extrinsic geometry of 2-D surfaces, hep-th preprint, 1994

[50] K. Viswanathan and R. Parthasarathy, Annals Phys., 206 (1991), 237-254

[51] Z. Wang, Phys. Lett. B, 264 (1991), 39-44

[52] T. Willmore, Riemannian geometry, Oxford Univ. Press, 1993

[53] K. Yoshida, Inter. Jour. Mod. Phys. A, 7 (1992), 4353-4375; Mod. Phys. Lett. A, 7 (1992), 4015-4038

[54] W. Zakrzewski, Low dimensional sigma models, Hilger, 1989

[55] R. Zucchini, Phys. Lett. B, 260 (1991), 296-302; Comm. Math. Phys., 152 (1993), 269-279

[56] R. Zucchini, Class. Quantum Gravity, 10 (1993), 253-278; 11 (1994), 1697-1724